

ION TEMPERATURE SENSITIVE EFFECT IN TRANSIENT LANGMUIR PROBE RESPONSE

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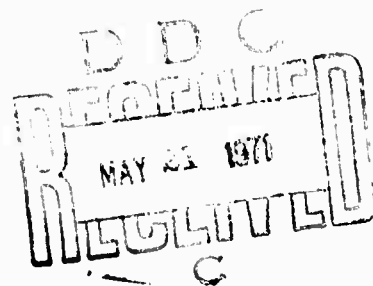
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ABSTRACT

A theory is presented for a method, recently proposed by Hester and Sonin, of determining the ion temperature in a plasma by measuring the transient current response of a highly negative, cylindrical Langmuir probe under conditions where collection is collision-free and the ratio of probe radius to Debye length is small. The ion component of the current does not approach its final, steady-state value monotonically, but exhibits a strong, ion temperature dependent overshoot in the first few ion-plasma periods following the biasing of the probe. Analytical formulae are derived for a Maxwellian plasma, and convenient graphical results are presented. The possible masking of the overshoot by a transient displacement current is discussed and found to be often negligible. For other conditions an alternative measuring method is suggested in which the contribution of the displacement current cancels out.

I - Introduction

Hester and Sonin¹ have recently proposed a method of determining the ion temperature in a plasma by measuring the transient current drawn by a long, cylindrical Langmuir probe to which a large, negative potential has been applied step-wise. They performed some computations showing that the ion current exhibits an initial overshoot which is very sensitive to the ion temperature when the probe radius is small compared with the Debye length. Such an effect would be of considerable interest for diagnostic purposes because steady-state probe response is, under all conditions, very weakly dependent on that plasma property².

In this paper, we present a theoretical formulation of that problem which yields both a simple analytical formula for the transient current and a clear understanding of the overshoot. The treatment is limited to the first ion-plasma period following the biasing of the probe, which is the region where the response is most sensitive to the ion temperature. The basic approach is discussed in the next section. In Sec. III we derive some needed results about the potential field set up by the probe. The phenomenon of an ion current overshoot is discussed in Sec. IV and in Sec. V a formula is derived for the current and convenient graphical results are presented. Finally, a discussion of experimental aspects of the method, and its usefulness, is given in Sec. VI. In particular, the possible masking of the overshoot by the displacement current, which is registered in the external

measuring circuitry together with the ion current, is discussed. It is shown that such masking is often negligible. For other conditions where it is not negligible, an alternative, pulsed probe method is suggested in which displacement effects cancel out.

II - Basic formulation

Consider the problem of a quiescent plasma and an infinitely long, cylindrical Langmuir probe of radius r_p to which a negative potential V_p (relative to the plasma) is applied at $t = 0$. It is desired to obtain the ion current to the probe for positive times. The potential field, V , is governed by Poisson's equation, which in nondimensional form reads

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} \psi = \epsilon^2 (n_i - n_e) \quad , \quad (1)$$

with boundary conditions

$$\psi(\rho = 1) = \psi_p \equiv -eV_p/kT_e \quad ,$$

$$\psi(\infty) = 0 \quad ,$$

where

$$\psi = -eV/kT_e \quad , \quad \rho = r/r_p \quad ,$$

$$\epsilon = r_p/\lambda_D \quad , \quad n_i = Z_i N_i/N_0 \quad , \quad n_e = N_e/N_0 \quad ;$$

N_e and N_i are the electron and ion densities, T_e and N_0 the unperturbed electron temperature and density, r is the radial distance to the probe axis, Z_i the ion charge number and λ_D the electron Debye length, $(\kappa T_e / 4\pi e^2 N_0)^{1/2}$. It will be here assumed that all mean free paths are large compared with λ_D , so that if $\epsilon \leq 0(1)$, current collection will be collision-free and n_e and n_i will be given by the corresponding time-dependent Vlasov equation. Simple dimensional considerations then show that

$$\frac{j(\tau \geq 0)}{j_\infty} = \frac{j}{j_\infty} \left(\tau, \epsilon, \psi_p, \beta, \frac{m_e}{m_i} \right) \quad (2)$$

where

$$\beta = T_i / Z_i T_e, \quad \tau = \omega_p t,$$

j is the ion current density, j_∞ the steady-state current density corresponding to the probe potential V_p , ω_p the ion-plasma frequency $(4\pi e^2 Z_i N_0 / m_i)^{1/2}$, m_e the electron mass, and m_i and T_i the ion mass and temperature respectively; Eq. (2) will also depend on the probe potential for negative times, if that is not zero. One expects $j/j_\infty \rightarrow 1$ as $\tau \rightarrow \infty$.

The general solution of the problem defined above is quite difficult. We are interested, however, in specific conditions under which j/j_∞ does not increase monotonically from $j(\tau = 0)/j_\infty$ to 1, but

exhibits a strong overshoot. It is quite fortunate that such conditions result also in a considerable simplification of the problem. We assume the following:

(a) The probe is at plasma potential at $\tau < 0$. The selection of any other initial probe potential would somewhat mask the influence of T_i on the overshoot, since this influence depends critically on the initial distribution function in the neighborhood of the probe. However, the above condition makes the analysis easier because of the simplicity of the initial plasma state; we have in particular²

$$\psi(\rho, \tau < 0) = 0 \quad ,$$

$$n_e(\rho, \tau \leq 0) = n_i(\rho, \tau \leq 0) = 1 - \pi^{-1} \sin^{-1} \rho^{-1} \quad , \quad (3)$$

$$j(\tau \leq 0) = e N_0 (2\pi)^{-1/2} (\kappa T_i / m_i)^{1/2} \quad . \quad (4)$$

(b) $\psi_p \gg 1$. This assumption allows us to neglect the electron current. Moreover, it leads to a simple expression for n_e ,

$$n_e = \exp(-\psi) \quad , \quad (5)$$

valid everywhere except in the neighborhood of the probe where n_e will be exponentially small anyway [see also condition (d) below]. The equilibrium with the field will be reached in a few electron-plasma

periods, so that the dependence of (2) on m_e/m_i will disappear for, say, $\tau > 0.1$.

(c) $\psi_p \gg \beta$, and (d) $\epsilon \ll 1$. These two conditions are found to be essential for the appearance of a large overshoot. Condition (d) also implies that steady-state current collection will be orbital motion limited, so that²

$$j_\infty = e N_0 2^{1/2} \pi^{-1} (-Z_1 e v_p / m_i)^{1/2}, \quad (6)$$

and, assuming the unperturbed plasma to be Maxwellian,²

$$n_1(\tau \rightarrow \infty) \rightarrow 1 - \frac{1}{\pi} \int_0^\infty ds \exp(-s) \sin^{-1} \left[\frac{1}{\rho} \left(\frac{\psi_p + \beta s}{\psi + \beta s} \right)^{1/2} \right]. \quad (7)$$

Furthermore, (d) greatly simplifies the determination of ψ , and contributes also to the validity of Eq. (5), since a region of order of r_p , where Eq. (5) fails, then contains a negligible amount of negative charge even if $n_e = 0(1)$.

All these conditions greatly simplify the analysis. First, it is possible to derive an accurate analytical expression for $\partial\psi/\partial\rho$ without solving simultaneously the ion Vlasov equation. That expression is only valid in a certain (large) neighborhood of the probe. This restricts the theory to times bound by an upper value τ_m , since for sufficiently long times the probe will be collecting ions that were initially outside that neighborhood. Fortunately, again, the overshoot occurs at shorter times, and is, in fact, most sensitive to T_i in the τ -range where the theory is valid.

Second, once the field is known ion trajectories may be determined explicitly. Moreover the current to the probe is now linear in the initial ion distribution function and this allows a further simplification. Let f_1 be the ion distribution function after integration over the velocities parallel to the probe. We shall assume that far away from the probe f_1 is isotropic

$$f_1(\rho \rightarrow \infty) = Z_1^{-1} N_0 g(v_1)$$

where v_1 is the ion speed in the ρ - ϕ plane (ϕ being a polar angle).

f_1 is normalized such that

$$\int_0^{\infty} 2\pi v_1 g(v_1) dv_1 = 1 \quad ;$$

for a Maxwellian plasma

$$g = \frac{m_1}{2\pi\kappa T_1} \exp\left(-\frac{m_1 v_1^2}{2\kappa T_1}\right) \quad . \quad (8)$$

Then we clearly have

$$f_1(\tau \leq 0) = Z_1^{-1} N_0 g(v_1) \quad , \quad |\gamma| > \sin^{-1} \rho^{-1} \quad (9a)$$

$$= 0 \quad , \quad |\gamma| < \sin^{-1} \rho^{-1} \quad (9b)$$

where γ is a velocity-vector angle ($-\pi \leq \gamma \leq \pi$); this is illustrated in Fig. 1. For any g , Eqs. (9) result in the initial density given by Eq. (3); the ion depletion is due to the particle loss caused by thermal motion. Now let j^* be the current density obtained by assuming an initial ion distribution function with all ions having velocities with the same direction and magnitude, v_1^* ; see Fig. 2, where the polar axis has been chosen parallel to the velocity for simplicity. (We note that in the shaded area there are no ions while outside it the density is uniform, N_0/Z_1 .) Then one easily concludes from the symmetry of the problem that j may be written as

$$j(\tau \geq 0) = \int_0^\infty 2\pi v_1^* g(v_1^*) dv_1^* j^*(v_1^*, \tau \geq 0) \quad (10)$$

Finally the determination of j^* itself may be simplified by using the following approach. The non-shaded region of Fig. 2 may be divided in two mutually exclusive parts, A^* and B^* . A point $o-\phi$ belongs to A^* if an ion having such initial coordinates will hit the probe at some time $\tau > 0$. Computing the time of flight of an A^* -ion to the probe we may then define a function $a^*(\tau)$ that represents the area of region A^* that has been "collected" by the time τ . From the uniformity of the density it follows that

$$2\pi r_p j^* = eN_0 da^*/d\tau$$

or

$$\frac{j^*}{j_\infty} = \frac{\epsilon}{(8\psi_p)^{1/2}} \frac{d\hat{a}^*}{d\tau}$$

where \hat{a}^* is a nondimensional area, $\hat{a}^* = a^*/r_p^2$. For the time-averaged current we have

$$\frac{\bar{j}^*}{j_\infty} = \frac{1}{\tau} \int_0^\tau \frac{j^*(\tau')}{j_\infty} d\tau' = \frac{\epsilon \hat{a}^*}{(8\psi_p)^{1/2} \tau} \quad (11)$$

In the next section we will find the aforementioned expression for the electric field, $\partial\psi/\partial\rho$. In Sec. IV we will determine the boundaries of the region A^* and discuss the overshoot phenomenon. In Sec. V we will compute the area \hat{a}^* and from there the current to the probe.

III - The potential field

In the next two sections we shall have to integrate the electric field $\partial\psi/\partial\rho$ along the trajectory of a particle to obtain its velocity and its time of flight to the probe. A formal expression for $\partial\psi/\partial\rho$ can be obtained from Eq. (1),

$$\frac{\partial\psi}{\partial\rho} = \frac{-\psi_p}{\rho} \frac{\delta(\tau)}{\rho} + \frac{\epsilon^2}{\rho} \int_1^\rho \rho' d\rho' v(\rho', \tau) \quad (12)$$

where

$$\delta \equiv -\psi_p^{-1} \partial\psi/\partial\rho \Big|_{\rho=1}, \quad v \equiv n_i - n_e ;$$

Eq. (12) may be rewritten as

$$\frac{\partial \psi}{\partial \rho} = -\frac{\psi_p}{\rho} \frac{\delta(\tau)}{\rho} + \frac{\epsilon^2}{\rho} \frac{\rho^2 - 1}{2} \langle v(\rho, \tau) \rangle, \quad (13)$$

$\langle v \rangle$ being given by

$$(\rho^2 - 1) \langle v \rangle = \int_1^{\rho^2} v d(\rho'^2). \quad (14)$$

We note that δ is proportional to the electric field at the probe.

Its relation to the overall charge density follows easily from (12):

$$\delta = \psi_p^{-1} \epsilon^2 \int_1^{\infty} \rho d\rho v = \left(\int_1^{\infty} \rho d\rho v \right) / \left(\int_1^{\infty} \rho d\rho v \ln \rho \right). \quad (15)$$

Thus δ^{-1} represents the logarithm of a characteristic shielding distance.

To carry out the aforementioned integrations we shall substitute constants $\bar{\delta}$ and \bar{v} for $\delta(\tau)$ and $\langle v(\rho, \tau) \rangle$ in Eq. (13). Our purpose now is to show that δ and (with some restrictions on ρ) $\langle v \rangle$ vary little, and to find close bounds for them; this will allow us to choose appropriate values for $\bar{\delta}$ and \bar{v} , and to determine the error in the approximations used. The basic approach will be to study the limits $\tau \rightarrow 0$ and $\tau \rightarrow \infty$, for which v is known explicitly [see Eqs. (3), (5) and (7)].

(A) Bounds for δ at $\tau \approx 0^3$ and $\tau \rightarrow \infty$.

We must first establish the following theorem: Let $\psi_k(\rho)$ be defined by

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} \psi_k = F_k[\psi_k(\rho), \rho] \quad \rho \geq 1, \quad (16)$$

$$\psi_k(1) = \psi_p, \quad \psi_k(\infty) = 0 \quad (17)$$

where $k = 1, 2$ and F_k is bounded, compatible with Eqs. (17), and such that ψ_k is continuous and monotonic. Let

$$F_2[f, \rho] \geq F_1[f, \rho] \quad \rho \geq 1, \quad \psi_p \geq f \geq 0, \quad (18)$$

and $F_1[f, \rho]$ be nondecreasing in f . Then, if $\delta_k \equiv -\psi_p^{-1} \frac{\partial \psi_k}{\partial \rho} \Big|_{\rho=1}$,

$$\delta_2 \geq \delta_1.$$

To prove this let us assume that $\delta_2 < \delta_1$. Integrating (16) twice we get

$$\psi_k = \psi_p [1 - \delta_k \ln \rho] + \int_1^\rho \rho' d\rho' \ln \frac{\rho}{\rho'} F[\psi_k(\rho'), \rho'].$$

Thus in a certain neighborhood of 1, we have $\psi_2 > \psi_1$. Let s be the minimum value of ρ such that $\psi_2(s) = \psi_1(s)$; such value exists, whether finite or infinite, because $\psi_2(\infty) = \psi_1(\infty)$. It follows that

$$\begin{aligned}
 0 &= \psi_1(s) - \psi_2(s) = \psi_p(\delta_2 - \delta_1) \ln s + \int_1^s \rho \, d\rho \ln \frac{s}{\rho} \\
 &\cdot \{F_1[\psi_1(\rho), \rho] - F_2[\psi_2(\rho), \rho]\} \leq \psi_p(\delta_1 - \delta_2) \ln s \\
 &+ \int_1^s \rho \, d\rho \ln \frac{s}{\rho} \{F_1[\psi_1(\rho), \rho] - F_1[\psi_2(\rho), \rho]\} < 0 \quad ,
 \end{aligned}$$

proving that δ_2 cannot be less than δ_1 .

Upper and lower bounds for $\delta(\tau \approx 0)$ can now be found. The right-hand side of Eq. (1) is then, according to Eqs. (3) and (5),

$$F = \epsilon^2 [1 - \exp(-\psi)] - \frac{\epsilon^2}{\pi} \sin^{-1} \frac{1}{\rho} \quad .$$

Introducing

$$\begin{aligned}
 F_2^{(1)}[\psi_2^{(1)}(\rho), \rho] &= \epsilon^2 [1 - \exp(-\psi_p)] - \frac{\epsilon^2}{\pi} \sin^{-1} \frac{1}{\rho} \quad \psi_2^{(1)} \geq 1 - \exp(-\psi_p) \quad , \\
 &= \epsilon^2 \psi_2^{(1)} - \frac{\epsilon^2}{\pi} \sin^{-1} \frac{1}{\rho} \quad \psi_2^{(1)} \leq 1 - \exp(-\psi_p) \quad ,
 \end{aligned}$$

our theorem can be seen to apply if $F \rightarrow F_1$, $\psi \rightarrow \psi_1$ and $F_2^{(1)} \rightarrow F_2$, $\psi_2^{(1)} \rightarrow \psi_2$. Therefore

$$\delta(\tau \approx 0) \leq \delta_2^{(1)} \quad . \quad (19)$$

Since $F_2^{(1)}$ is linear, $\delta_2^{(1)}$ is easily found; neglecting $\exp(-\psi_p)$, $\epsilon^2 \rho^{(1)}/\pi \psi_p$ and smaller terms in favor of unity, we get

$$1/\delta_2^{(1)} = [1 - \psi_p^{-1} (1 - \epsilon^2 \rho_{(1)}^2/4)] \ln \rho_{(1)} \quad . \quad (20a)$$

$\rho_{(1)}$ is such that $\psi_2^{(1)}[\rho_{(1)}] = 1 - \exp(-\psi_p)$ and is given by

$$\begin{aligned} \psi_p = 1 - \frac{\epsilon^2 \rho_{(1)}^2}{4} + \left[\frac{\epsilon^2 \rho_{(1)}^2}{2} + \epsilon \rho_{(1)} \frac{K_1(\epsilon \rho_{(1)})}{K_0(\epsilon \rho_{(1)})} \right. \\ \left. - \frac{\epsilon^2}{\pi} \left\{ \rho_{(1)} + \int_{\rho_{(1)}}^{\infty} \frac{K_0(\epsilon \rho) d\rho}{K_0(\epsilon \rho_{(1)})} \right\} \right] \ln \rho_{(1)} \quad . \quad (20b) \end{aligned}$$

For a lower bound, let us introduce

$$\begin{aligned} F_1^{(2)}[\psi_1^{(2)}(\rho), \rho] = \epsilon^2 [1 - \exp(-p)] - \frac{\epsilon^2}{\pi} \sin^{-1} \frac{1}{\rho} \quad \psi_1^{(2)} \geq p \quad , \\ = \epsilon^2 \frac{1 - \exp(-p)}{p} \psi_1^{(2)} - \frac{\epsilon^2}{\pi} \sin^{-1} \frac{1}{\rho} \quad \psi_1^{(2)} \leq p \quad , \end{aligned}$$

where we wrote p for $1 - \exp(-\psi_p)$ to simplify the equations. Our theorem applies again with $F \rightarrow F_2$, $\psi \rightarrow \psi_2$ and $F_1^{(2)} \rightarrow F_1$, $\psi_1^{(2)} \rightarrow \psi_1$; thus

$$\delta(\tau \approx 0) \geq \delta_1^{(2)} \quad . \quad (21)$$

$\delta_1^{(2)}$ is given by

$$1/\delta_1^{(2)} = [1 - \psi_p^{-1} (1 - \epsilon^2 \rho_{(2)}^2/4)] [\ln \rho_{(2)} + 0.23] \quad , \quad (22a)$$

$$\psi_p = 1 - \frac{\epsilon^2 \rho^2(2)}{4} + \left[\frac{\epsilon^2 \rho^2(2)}{2} + \epsilon \rho(2) \frac{K_1(\epsilon \rho(2))}{K_0(\epsilon \rho(2))} - \frac{\epsilon^2}{\pi} 1.26 \rho(2) + \int_{\rho(2)}^{\infty} \frac{K_0(\epsilon \rho)}{K_0(\epsilon \rho(2))} d\rho \right] [\ln \rho(2) + 0.23] \quad (22b)$$

For $\delta(\tau = \infty)$, we have [according to (7)]

$$F = \epsilon^2 [1 - \exp(-\psi)] - \frac{\epsilon^2}{\pi} \int_0^{\infty} ds \exp(-s) \sin^{-1} \left[\frac{1}{\rho} \left(\frac{\psi_p + \beta s}{\psi + \beta s} \right)^{1/2} \right] \quad (23)$$

as the right-hand side of Eq. (1). An upper bound follows immediately:

$$\delta(\tau = \infty) \leq \delta(\tau \approx 0) \leq \delta_2^{(1)} \quad (24)$$

because $\partial F / \partial \psi > 0$ in (23) and

$$\frac{1}{\pi} \int_0^{\infty} ds \exp(-s) \sin^{-1} \left[\frac{1}{\rho} \left(\frac{\psi_p + \beta s}{\psi + \beta s} \right)^{1/2} \right] \geq \frac{1}{\pi} \sin^{-1} \frac{1}{\rho}$$

for $0 \leq \psi \leq \psi_p$. For a lower bound, we note that for $\bar{\beta} > \bar{\beta}$

$$\delta(\tau = \infty, \bar{\beta}) \geq \delta(\tau = \infty, \bar{\beta})$$

since

$$\frac{\psi_p + \bar{\beta} s}{\psi + \bar{\beta} s} \leq \frac{\psi_p + \bar{\beta} s}{\psi + \bar{\beta} s}$$

for $s > 0$ and $0 \leq \psi \leq \psi_p$; thus, it suffices to find a lower bound for $\delta(\tau = \infty, \beta = 0)$. Let us introduce

$$F_1^{(3)} = \epsilon^2 [1 - \exp(-p)] - \frac{\epsilon^2}{\pi} \int_0^\infty ds \exp(-s) \left[\frac{\pi}{2}, \sin^{-1} \left\{ \frac{1}{\rho} \cdot \left(\frac{\psi_p + \beta s}{p + \beta s} \right)^{1/2} \right\} \right] \psi_1^{(3)} \geq p, \quad (25a)$$

$$= \epsilon^2 \frac{1 - \exp(-p)}{p} \psi_1^{(3)} - \frac{\epsilon^2}{\pi} \int_0^\infty ds \exp(-s) \left[\frac{\pi}{2}, \sin^{-1} \frac{1}{\rho} \cdot \left(\frac{\psi_p + \beta s}{(\rho_{(3)}^2/\rho^2) + \beta s} \right)^{1/2} \right] \psi_1^{(3)} \leq p, \quad (25b)$$

where again $p = 1 - \exp(-\psi_p)$, and in the brackets inside the integrals above, the second terms are to be used whenever the arguments of \sin^{-1} are less than 1; otherwise $\pi/2$ should be used. Notice that we have retained a finite β : if we take $\beta \rightarrow 0$ in (25), $\psi_1^{(3)}$ cannot be made to vanish at infinity. However if we first retain $\beta \neq 0$, solve for $\psi_1^{(3)}$ and $\delta_1^{(3)}$, and then let $\beta \rightarrow 0$, we find a finite $\delta_1^{(3)}$; the singularity of this limit is related to well known differences between the steady-state currents to a negative cylindrical probe under condition $T_1 = 0$ and condition $T_1 \rightarrow 0$.³ We also note that $F_1^{(3)}$ is not a point function but a functional of $\psi_1^{(3)}$, because of the appearance in Eq. (25) of $\rho_{(3)}$, the point where $\psi_1^{(3)} = p$; making $F \rightarrow F_2$, $\psi \rightarrow \psi_2$ and $F_1^{(3)} \rightarrow F_1$, $\psi_1^{(3)} \rightarrow \psi_1$, one can see that (18) is nevertheless satisfied⁴ and that

F_1 , i.e., $F_1^{(3)}$, satisfies a certain nondecreasing condition: if $\psi_1^{(3)}(\rho') \leq \psi(\rho')$ in the entire range $1 \leq \rho' \leq \rho$, $F_1^{(3)}[\psi_1^{(3)}, \rho] \leq F_1^{(3)}[\psi, \rho]$.

Accounting for this, our theorem may be seen to apply again, giving

$$\delta(\tau = \infty, \beta) \geq \delta(\tau = \infty, \beta = 0) \geq \delta_1^{(3)}, \quad (26)$$

$$1/\delta_1^{(3)} = [1 - \psi_p^{-1} \{ 1 - \epsilon^2 R^2/4 + \epsilon^2 R \psi_p^{1/2} 1.26/\pi \}] \cdot [\ln R + 0.23] \quad (27a)$$

$$\psi_p = 1 - \frac{\epsilon^2 R^2}{4} + \left\{ \frac{\epsilon^2 R^2}{2} + \epsilon R \frac{K_1(\epsilon R)}{K_0(\epsilon R)} - \epsilon^2 \psi_p^{1/2} \frac{1.26}{\pi} \right. \\ \left. \cdot \left\{ R + \frac{K_1(\epsilon R)}{\epsilon K_0(\epsilon R)} \right\} (\ln R + 0.23) + \epsilon^2 \psi_p^{1/2} R \frac{1.26}{\pi} \right\}, \quad (27b)$$

where $R = \rho_{(3)}/1.26$. We have neglected $\exp(-\psi_p)$ and $\epsilon^2 \rho_{(3)}/\pi \psi_p$, as in Eqs. (20) and (22), and also $\psi_p/3\rho_{(3)}^2$.

In the parameter range later considered,

$$10^{-2} \leq \epsilon \leq 10^{-1}, \quad 10 \leq \psi_p \leq 10^2, \quad (28)$$

the terms neglected in deriving Eqs. (20), (22) and (27) result in an error of less than 1%. Within such an error we also find

$1/\delta_1^{(2)} = 1/\delta_2^{(1)} + 0.23$, and, within an error of less than 1% for $\epsilon = 10^{-2}$ and less than 2.5% for $\epsilon = 10^{-1}$, we find $\delta_1^{(3)} = \delta_1^{(2)}$.

(B) Bounds for $\langle v \rangle$ at $\tau \approx 0$ and $\tau \rightarrow \infty$.

For $\tau \approx 0$ we have

$$v = 1 - \pi^{-1} \sin^{-1} \rho^{-1} - \exp(-\psi) \quad (29)$$

Since $\psi_p \gg 1$, we get $v(1) \approx 1/2$. As ρ increases v approaches unity and remains close to it until ψ becomes small enough for the last term in Eq. (29) to be important. At $\rho = 1.05$, $v > 0.60$. At $\rho = \rho_{(1)}$ as given by Eq. (20b), $\psi \approx 1$ since at such distances $\psi \approx \psi_2^{(1)}$; thus $v[\rho_{(1)}] \approx 0.63$. Therefore $v > 0.60$ between $\rho = 1.05$ and ρ slightly larger than $\rho_{(1)}$; over most of this range v will be close to 1. Now, substitute a constant $\bar{\delta}$ for $\delta(\tau)$ in Eq. (14) [see Eq. (35) below], and define

$$\rho_m = (2\psi_p \bar{\delta})^{1/2} \epsilon^{-1} \quad (30)$$

We have

$$\frac{\partial \psi}{\partial \rho} = \frac{-\psi_p \bar{\delta}}{\rho} \left[1 - \langle v \rangle \frac{\rho^2}{\rho_m^2} + \frac{\langle v \rangle}{\rho_m^2} \right]$$

For $\psi_p \gg 1$, $\epsilon \ll 1$, ρ_m^2 is quite large so that the last term in the bracket can be neglected. The obvious requirement $\partial \psi / \partial \rho < 0$ then yields

$$\langle v \rangle \rho^2 / \rho_m^2 < 1 \quad ;$$

this shows that $\langle v \rangle$ decays rapidly for $\rho > \rho_m$. On the other hand one finds that $\rho_{(1)}$ is always about $3/4 \rho_m$. This, together with Eq. (14) and the conclusions derived above for the range $1.05 < \rho < \rho_{(1)}$ indicate that it is safe to assume that

$$0.60 \leq \langle v \rangle \leq 1 \quad (31)$$

as long as $\rho \leq \rho_m$.

In the limit $\tau \rightarrow \infty$ the increase of n_1 with ρ is not as fast as in the $\tau \approx 0$ case [compare Eqs. (3) and (7)]. On the other hand, ψ has a slower decay because $\delta(\tau \rightarrow \infty) \leq \delta(\tau \approx 0)$, so that n_e remains negligible over a larger ρ -range. At $\rho = \rho_{(3)}$, $\psi \approx 1$; although $v(\rho_{(3)})$ may be as low as 0.58 (because $n_1 < 1$) we have $\rho_{(3)} > \rho_{(1)}$, that is, $\rho_{(3)}$ is closer to ρ_m . For the lowest ρ values, we still have $v(\rho = 1.05) = 0.60$. It appears that the inequalities in (31) are still valid as long as $\rho \leq \rho_m$.

(c) Bounds for arbitrary τ .

According to Eqs. (3) and (7), we have $n_1(\tau \approx 0) > n_1(\tau \rightarrow \infty)$. There is therefore a net decrease in the ion density in the neighborhood of the probe. Let us assume that this ion depletion is monotonic in time. Then

$$\delta(\tau \approx 0) \geq \delta(\tau) \geq \delta(\tau \rightarrow \infty) \quad (32)$$

since δ goes down with the ion density. It also follows that for any τ

$$0.60 < \langle v \rangle < 1 \quad , \quad \text{for } \rho \leq \rho_m \quad . \quad (33)$$

We should point out that inequalities (32) and (33) do not actually require that n_1 be monotonically decreasing in time but that the weaker condition

$$n_1(\tau \approx 0) \geq n_1(\tau) \geq n_1(\tau \rightarrow \infty) \quad (34)$$

be satisfied. In fact, since both δ and $\langle v \rangle$ depend on n_1 in a global way only, condition (34) could be locally violated without invalidating (32) or (33). The assumption $\partial n_1 / \partial \tau < 0$ appears well justified at least in a global way since the current density j exceeds j_∞ quite soon and remains that way thereafter.

We now define

$$1/\bar{\delta} \equiv (1 + 0.63\epsilon^2 \rho_{(1)}^2 / \pi \psi_p^{1/2}) / \delta_2^{(1)} + 0.12 \quad (35a)$$

and introduce

$$Y(\epsilon, \psi_p) \equiv 1/\bar{\delta} - \ln \epsilon^{-1} \quad ; \quad (35b)$$

Y is given graphically in Fig. 3. We shall use $\bar{\delta}$ in place of $\delta(\tau)$ in Eq. (13). The maximum possible error in equating $\delta(\tau)$ to $\bar{\delta}$ goes down with increasing ψ_p and ϵ^{-1} . We find that the error is

less than 5% for ϵ^{-1} , $\psi_p = 10$ and less than 1% for $\epsilon = 10^{-2}$, $\psi_p \geq 10$. This estimate is conservative because the lower bound for $\delta(\tau)$ has been obtained from the limit $\tau \rightarrow \infty$ while only a limited τ -range will be considered. Moreover $\partial\psi/\partial\rho$ will only be used under integral signs, leading to a smaller error.

We shall also use a constant value \bar{v} instead of $\langle v \rangle$ in Eq. (13). It follows from (33) that putting $\bar{v} = 0.80$ produces a maximum possible error of ± 0.20 . It should be noted that $\langle v \rangle$ will only appear inside integrals between $\rho = 1$ and $\rho \leq \rho_m$; this will obviously result in a more typical error for $\langle v \rangle \approx \bar{v}$, of ± 0.10 . The errors that the approximations for δ and $\langle v \rangle$ produce in the determination of the ion temperature will be discussed in Sec. VI.

The inequality $\rho < \rho_m$ will appear in other parts of our analysis as a condition for the accuracy of the theory. It is not a very precise condition in the sense that values of ρ slightly larger than ρ_m may be considered, since, as just indicated, integration over $\partial\psi/\partial\rho$ reduces the error in $\langle v \rangle \approx \bar{v}$. ρ_m may be considered as a characteristic shielding distance since it is close to $\rho_{(1)}$, which is itself very close to $\exp \bar{\delta}^{-1}$, and, moreover, for $\rho > \rho_m$ the plasma becomes rapidly quasineutral and both potential and electric field become small. It may be illustrative to note that assuming an idealized charge density $v = 1$ up to a certain distance ρ_F , and $v = 0$ for $\rho > \rho_F$, the first two members of Eq. (15) give $\rho_F = \rho_m$. The first and last members then give

$$\delta = (\ln \rho_m - 1/2)^{-1},$$

which can be used as an approximate equation for $\bar{\delta}$,

$$\bar{\delta} = 2(\ln \psi_p + \ln \bar{\delta} - 2 \ln \varepsilon + \ln 2 - 1)^{-1}. \quad (36)$$

IV - The overshoot effect

To determine the region A* in Fig. 2, consider the radial motion of an ion,

$$m_1 \frac{d^2 r}{dt^2} = -Z_1 e \frac{\partial V}{\partial r} + \frac{L^2}{m_1 r^3}. \quad (37)$$

The angular momentum is conserved so that if r_0, ϕ_0 are the initial coordinates of the ion, $L = m_1 v_1^* r_0 \sin \phi_0$. In non-dimensional form, Eq. (37) reads

$$\varepsilon^2 \frac{d^2 \rho}{d\tau^2} = \frac{\partial \psi}{\partial \rho} + 2\beta^* \sin^2 \phi_0 \frac{\rho_0^2}{\rho^3} \quad (38)$$

where $\beta^* = m_1 (v_1^*)^2 / 2Z_1 \kappa T_e$; from Sec. III, $\partial \psi / \partial \rho = -\psi_p \bar{\delta}^{-1} (1 - \bar{v}_0^2 \rho_m^{-2})$. Integrating (38) we get

$$\begin{aligned} \frac{1}{2} \left(\varepsilon \frac{d\rho}{d\tau} \right)^2 = & -\psi_p \bar{\delta} \ln \frac{\rho}{\rho_0} + \beta^* (1 - \sin^2 \phi_0 \frac{\rho_0^2}{\rho^2}) \\ & - \varepsilon^2 \frac{\bar{v}}{4} (\rho_0^2 - \rho^2). \end{aligned} \quad (39)$$

For (ρ_0, ϕ_0) to belong to A^* , the right-hand side of Eq. (39) must be non-negative at $\rho = 1$,

$$\beta^* \rho_0^2 \sin^2 \phi_0 \leq \beta^* + \psi_p \bar{\delta} \ln \rho_0 - \epsilon^2 \bar{v} (\rho_0^2 - 1)/4 .$$

This leads to

$$|\sin \phi_0| \leq G(\rho_0)/\rho_0 , \quad (40a)$$

$$G \equiv [1 + \ln \rho_0^2 / \alpha^* - \bar{v} (\rho_0^2 - 1) / \alpha^* \epsilon^2]^{1/2} \quad (40b)$$

where $\alpha^* \equiv 2\beta^* / \psi_p \bar{\delta}$.

It is interesting to note how A^* changes with α^* . For $\alpha^* > 1$ we have $G(\rho_0)/\rho_0 \leq 1$ for all $\rho_0 \geq 1$. For $\alpha^* < 1$, however, there is a value $\rho_q > 1$ such that $G(\rho_q)/\rho_q = 1$; then $G(\rho_0)/\rho_0 > 1$ for $\rho_0 < \rho_q$. As shown below, the regime of interest is $\alpha^* \ll 1$; condition (c) (Sec. II) ensures that this is satisfied here. The upper boundary of A^* , $\rho_A(\phi)$, is then given by

$$|\sin \phi| = \min\{1, G(\bar{\rho}_A)/\bar{\rho}_A\} , \quad (41)$$

and is schematically represented in Fig. 2. As α^* increases point q moves down, reaching $\rho = 1$ at $\alpha^* = 1$.

The preceding analysis shows that the region A^* is determined by considerations of angular momentum at $\tau = 0$. As pointed out in

Ref. 1 the lower the ion temperature, the farther from the probe one must go to find ions at $\tau = 0$ that will miss the probe due to their angular momentum; in terms of Fig. 2, when β^* decreases point q and the whole boundary of Λ^* moves upward. We will now show that \bar{j}^*/j_∞ may exhibit a significant overshoot; a qualitative, but detailed, picture of this phenomenon can be given on the basis of Fig. 2. The overshoot may be seen as caused by the sudden set-up of the potential field which traps low angular momentum ions in the neighborhood of the probe; in the steady state corresponding to a probe potential ψ_p , most of that neighborhood would be populated by high angular momentum ions.

Up to a certain distance from the probe (say $\rho < r_m/3$), $\partial\psi/\partial\rho \sim \rho^{-1}$. For ρ_0 in that region, the time of flight to the probe $\tau_0(\rho_0, \phi_0)$ grows (roughly) linearly with ρ_0 so that, according to Fig. 2, $\hat{a}^* \sim \tau^2$ (except for $\tau \ll 1$) as long as $\rho_0 < \rho_q$. More precisely, one finds

$$\hat{a}^* \sim \rho_m^2 \tau^2 \sim \psi_p \bar{\delta\tau}^2 / \epsilon^2$$

and therefore

$$\bar{j}^*/j_\infty \sim \epsilon \hat{a}^*/\tau \psi_p^{1/2} \sim \psi_p^{1/2} \bar{\delta\tau}/\epsilon \quad ; \quad (42)$$

\bar{j}^*/j_∞ increases with both ψ_p and ϵ^{-1} . For $\tau > \tau_q \equiv \tau_0(\rho_0 = \rho_q, \phi_0 = \pi/2)$, $d\hat{a}^*/d\tau$ will be nearly a constant, because \hat{a}^* is almost

linear in ρ_0 ; thus \bar{j}^*/j_∞ will taper off. For even larger τ , particles from outside the region where $\partial\psi/\partial\rho \sim \rho^{-1}$ will begin to be collected; the field now decreases rapidly and so does the mean velocity of the ion, ρ_0/τ_0 . Therefore $d\hat{a}^*/d\tau$ will decrease, and \bar{j}^*/j_∞ will peak and begin to fall. As $\tau \rightarrow \infty$, \bar{j}^*/j_∞ will eventually approach unity. In effect, the asymptotic value of $\bar{\rho}_A(\phi) \sin \phi$ in Fig. 2 is obviously

$$\bar{\rho}_A \sin \phi \rightarrow (\psi_p/\beta^*)^{1/2} \quad (43)$$

and as $\rho_0 \rightarrow \infty$ the average velocity of the ion becomes v_1^* , so that

$$\lim_{t \rightarrow \infty} \frac{da^*}{dt} = 2v_1^* r_p \left[\frac{-2Z_1 eV}{m_1 (v_1^*)^2} p \right]^{1/2} \quad (44)$$

and $\lim_{t \rightarrow \infty} j^* = j_\infty$.

The peak in \bar{j}^*/j_∞ is sensitive to the ion temperature. An increase of β^* , and therefore α^* , produces a decrease of ρ_q ; thus the linear growth of \bar{j}^*/j_∞ given in (42) is interrupted at a shorter time and this results in a lower peak for \bar{j}^*/j_∞ . When α^* reaches unity, $\rho_q = 1$ so that \hat{a}^* never grows quadratically in τ . $\alpha^* > 1$ may be roughly considered as a critical condition for the overshoot; the numerical results of Ref. 1 (largest $\alpha^* \approx 0.70$) indicate that this is actually the case. For decreasing β^* , ρ_q increases and the peak of \bar{j}^*/j_∞ goes up. Eventually, however, a value of β^* is reached for which ρ_q moves out of the $\partial\psi/\partial\rho \sim \rho^{-1}$ region; the growth of \bar{j}^*/j_∞ is then

stopped by the rapidly decreasing mean velocity of an ion along its trajectory, and not by the fact that the boundary of A^* is reached. The effect is the same for all smaller β^* so that the current is now insensitive to the actual value of β^* . Since $\rho_m/3$ grows with ϵ^{-1} , it is clear that using increasingly smaller probes results in the possibility of measuring increasingly smaller ion temperatures. We note here that although ρ_m grows also with ψ_p , so does ρ_q so that a change in the probe potential has no effect on the range of temperatures that can be measured by the probe.

To determine the dependence of the current peak on the various parameters we put $\tau_q \sim \rho_q/\rho_m$ in Eq. (42); according to Eq. (41)

$$\rho_q \sim \left(\frac{\psi_p}{\beta^*}\right)^{1/2} \left(\ln \frac{\psi_p}{\beta^*}\right)^{1/2}$$

so that

$$\frac{j^*}{j_\infty} \sim \left(\frac{\psi_p}{\beta^*}\right)^{1/2} \bar{\delta} \left[\ln \left(\frac{\psi_p}{\beta^*}\right)\right]^{1/2} . \quad (45)$$

This shows that the peak is fairly insensitive to the value of ϵ ; this prediction is supported by the numerical results of Ref. 1. The current becomes insensitive to β^* when $\rho_q \sim \rho_m$, or $\tau_q \sim 1$; this roughly corresponds to a value of β^*

$$\beta^* \sim \epsilon^2 . \quad (46)$$

The maximum peak in the current is then

$$\bar{j}^*/j_\infty \sim \psi_p^{1/2} \bar{\delta}/\epsilon, \quad (47)$$

so that for a given ψ_p the ratio of maximum currents for two different values of ϵ , ϵ_1 and ϵ_2 , is $(\epsilon_2/\epsilon_1)(\ln \epsilon_2^{-1}/\ln \epsilon_1^{-1})$. This is also supported by the computations of Ref. 1.

We would like to conclude by noting that (a) these qualitative results can be obviously extended to the case of an initial ion Maxwellian distribution; (b) for $\bar{\rho}_A \leq \rho_m$ the maximum possible error in $\bar{\rho}_A(\phi)$ due to the error in putting $\langle v \rangle \sim \bar{v} \equiv 0.80$ is around 1%, and the one due to approximating δ by $\bar{\delta}$ is around 2%; (c) the inequality given by (40), actually, is not a sufficient condition for $\rho_0 - \phi_0$ to belong to A^* : For $\phi_0 \leq \pi/2$, the right-hand side of Eq. (39) must be positive for all ρ between ρ_0 and 1; in other words, no potential barriers should appear. Defining

$$x = \rho^2/\rho_0^2, \quad W = -\ln x - \bar{v} \rho_0^2(1-x)/\rho_m^2 + \alpha^*(1 - \sin^2 \phi_0/x)$$

that condition reads

$$W(x) > 0 \quad \text{for} \quad \rho_0^{-2} < x < 1 \quad \text{whenever} \quad W(\rho_0^{-2}) \geq 0. \quad (48)$$

This is indeed satisfied when $\rho_0 \leq \rho_m$. As ρ_0 increases a value is found, dependent on both α^* and ϕ_0 and slightly larger than ρ_m , beyond which (48) is violated. For those values, however, the use of \bar{v} for

$\langle v \rangle$ becomes invalid so that no clear conclusion can be reached on the matter of possible transient potential barriers for (roughly) $\rho_0 > \rho_m$. This possibility can not in principle be ruled out, because, for small τ , the somewhat high density in the neighborhood of the probe steepens the potential field.

For $\phi_0 > \pi/2$, the sufficient condition is that $W(x)$ be positive for $\rho_0^{-2} < x < 1$ and be zero at some $x > 1$. The first part of this condition is the same as (48). The second part means that the ion, which for $\phi_0 > \pi/2$ moves initially away from the probe, must turn back at some $\rho > \rho_0$; this condition yields a new boundary for Λ^* , $\bar{\rho}_A(\phi)$, which is also represented in Fig. 2. This boundary, however, plays no role in our analysis because it is found that $\tau_0[\bar{\rho}_A(\phi), \phi] > \tau_m$, where τ_m is the largest value of τ considered (see next section for value of τ_m).

V - Ion current computation

The derivation of a formula for j/j_∞ presents no essential difficulty but it is long and tedious. We will now give schematically the mathematical steps involved in the derivation; some additional details are given in the Appendix. We made extensive use of the assumptions $\epsilon \ll 1$, $\psi_p \gg 1$, $\alpha^* \ll 1$. The formula derived, however, is not just asymptotic; we have checked throughout the numerical errors of the various approximations or simplifications in the algebra. To this effect we specifically assumed

$$10^{-2} \leq \epsilon \leq 10^{-1} \quad , \quad 10 \leq \psi_p \leq 10^2 \quad , \quad \alpha^* \leq 10^{-1} \quad .$$

(a) Time of flight to the probe $\tau_0(\rho_0, \phi_0)$.

The first step is to obtain $\tau_0(\rho_0, \phi_0)$, the time of flight of an A^* -ion to the probe. From Eq. (39) we get

$$\tau_0 = \frac{\epsilon}{2^{1/2}} \int_1^{\rho_0} d\rho \left[\beta^* \left(1 - \frac{\rho_0^2}{\rho^2} \sin^2 \phi_0 \right) - \psi_p \bar{\delta} \ln \frac{\rho}{\rho_0} - \epsilon^2 \frac{\bar{v}}{4} (\rho_0^2 - \rho^2) \right]^{-1/2} \quad (49)$$

for $\phi_0 \leq \pi/2$ and

$$\tau_0 = \tau_0(\pi - \phi_0) + 2 \frac{\epsilon}{2^{1/2}} \int_{\rho_0}^{\rho_T} d\rho \left[\beta^* \left(1 - \frac{\rho_0^2}{\rho^2} \sin^2 \phi_0 \right) - \psi_p \bar{\delta} \ln \frac{\rho}{\rho_0} - \epsilon^2 \frac{\bar{v}}{4} (\rho_0^2 - \rho^2) \right]^{-1/2} \quad (50)$$

for $\pi/2 \leq \phi_0 \leq \pi$. ρ_T is the zero of the square root inside the integral and represents the distance at which the ion turns back, heading toward the probe.

From Eq. (49) one obtains

$$\tau_0 = \pi^{1/2} \frac{\rho_0}{\rho_m} \exp\left(\frac{\alpha_c^*}{2}\right) \frac{\text{Erf}(\ln \rho_0 + \alpha_c^*/2)^{1/2} - \text{Erf}(\alpha_c^*/2)^{1/2}}{[1 - \bar{v} g_1(\alpha_c^*) \rho_0^2 / \rho_m^2 - \alpha^* \sin^2 \phi_0]^{1/2}} \quad (51)$$

where $\alpha_c^* = \alpha^* \cos^2 \phi_0$ and

$$g_1(\alpha_c^*) = 3^{1/2} \exp(\alpha_c^*) \frac{\text{Erfc}(3\alpha_c^*/2)^{1/2}}{\text{Erfc}(\alpha_c^*/2)^{1/2}} - 1. \quad (52)$$

From Eq. (50) one obtains

$$\tau_0 = \tau_0(\pi - \phi_0) + \frac{\rho_0}{\rho_m} \frac{2^{3/2} (\alpha_c^*)^{1/2}}{1 - \alpha^* - \frac{2}{v\rho_0/\rho_m^2}} \left[1 - \frac{2}{3} \alpha_c^* \frac{1 - 2\frac{2}{v\rho_0/\rho_m^2}}{(1 - \alpha_c^* - \frac{2}{v\rho_0/\rho_m^2})^2} \right]. \quad (53)$$

See the Appendix for details.

(b) Explicit expression for $\rho_0(\phi_0, \tau_0)$

Rearranging Eq. (51) we get (for $\phi_0 \leq \pi/2$)

$$\rho_0^2 = \frac{\rho_m^2 \tau_0^2 \exp(-\alpha_c^*) (1 - \alpha^* \sin^2 \phi_0)}{\pi \{ \text{Erf}(\ln \rho_0 + \alpha_c^*/2)^{1/2} - \text{Erf}(\alpha_c^*/2)^{1/2} \}^2} \cdot \left\{ 1 + \frac{2}{\pi} \tau^2 g_2(\alpha_c^*) \left[1 - \frac{\text{Erfc}(\ln \rho_0 + \alpha_c^*/2)^{1/2}}{\text{Erfc}(\alpha_c^*/2)^{1/2}} \right]^{-2} \right\}^{-1} \quad (54)$$

where

$$g_2(\alpha_c^*) = g_1(\alpha_c^*) \exp(-\alpha_c^*) \{ \text{Erfc}(\alpha_c^*/2)^{1/2} \}^{-2}. \quad (55)$$

The function g_2 remains remarkably constant,

$$0.732 = g_2(0) \leq g_2(\alpha_c^*) \leq g_2(0.1) = 0.782.$$

We simplify Eq. (54) by approximating $\exp(-\alpha_c^*) (1 - \alpha^* \sin^2 \phi_0)$ by $1 - \alpha^*$, g_2 by a constant \bar{g}_2 , and the bracket by unity. The first approximation has an error of less than 1%. The effect of the second one in the evaluation of the ion temperature will be later discussed;

we choose $\bar{g}_2 = 0.75$. The third approximation has a noticeable error for $\rho_0 = 0(1)$ only; because of the large value of ρ_m , $\rho_0 = 0(1)$ corresponds to τ_0 small, in which case the term with the bracket is entirely negligible. We get

$$\rho_0^2 [\text{Erf}(\ln \rho_0 + \alpha_c^*/2)^{1/2} - \text{Erf}(\alpha_c^*/2)^{1/2}]^2 = \rho_m^2 \tau_0^2 (1 - \alpha^*) (\pi + \bar{g}_2 \bar{v} \tau_0^2)^{-1} . \quad (56)$$

Rearranging Eq. (53), and using the same approximations leading to Eq. (56), we get (for $\pi/2 \leq \phi_0 \leq \pi$)

$$\rho_0^2 [\text{Erf}(\ln \rho_0 + \alpha_c^*/2)^{1/2} + \text{Erf}(\alpha_c^*/2)^{1/2}]^2 = \rho_m^2 \tau_0^2 (1 - \alpha^*) (\pi + \bar{g}_2 \bar{v} \tau_0^2)^{-1} \cdot (1 - \Delta)^{-1} \quad (57)$$

where

$$\Delta = 4 \text{ Erf}(\frac{\alpha_c^*}{2})^{1/2} (1 + \frac{\bar{g}_2 \bar{v} \tau_0^2}{\pi})^{-1} [\text{Erf}(\ln \rho_0 + \frac{\alpha_c^*}{2})^{1/2} + \text{Erf}(\frac{\alpha_c^*}{2})^{1/2}]^{-2} \cdot [\frac{\bar{v} \bar{g}_2 \tau_0^2}{\pi} \text{ Erf}(\ln \rho_0 + \frac{\alpha_c^*}{2})^{1/2} - \Gamma \{1 + \text{Erf}(\frac{\alpha_c^*}{2})^{1/2}\} - \Gamma^2 \text{ Erf}(\frac{\alpha_c^*}{2})^{1/2}]$$

and Γ is given by

$$\Gamma = \frac{[1 - \alpha^* - \bar{v} \bar{g}_1(\alpha_c^*) \rho_0^2 / \rho_m^2]^{1/2}}{1 - \alpha^* - \bar{v} \rho_0^2 / \rho_m^2} \left[1 - \frac{2}{3} \frac{\alpha_c^* (1 - 2\bar{v} \rho_0^2 / \rho_m^2)}{(1 - \alpha^* - \bar{v} \rho_0^2 / \rho_m^2)^2} - \frac{\alpha_c^*}{6} \right] - 1 .$$

Δ is found to be at most about 0.03 and will be now neglected. This will affect the value of $\hat{\alpha}^*$, to be computed below, in little more of 1% since Δ appears only in half of the ϕ_0 range.

Equations (56) and (57) may be now substituted by a single equation valid for all ϕ_0

$$\rho_0 [1 - (\frac{2\alpha^*}{\pi})^{1/2} \cos \phi_0] [1 - \frac{\text{Erfc}(\ln \rho_0 + \alpha_c^*/2)^{1/2}}{1 - (2\alpha^*/\pi)^{1/2} \cos \phi_0}] = \frac{\rho_m \tau_0 (1 - \alpha^*)^{1/2}}{(\pi + \sqrt{g_2} \tau_0^2)^{1/2}} ; \quad (58)$$

we have expanded $\text{Erfc}(\alpha_c^*/2)^{1/2}$ for small $(\alpha_c^*/2)^{1/2}$ with an error of less than 1%.

We now approximate the second bracket in Eq. (58) by

$$1 - \text{Erfc}[\ln \rho_0 \frac{1 - (2\alpha^*/\pi)^{1/2} \cos \phi_0}{(1 - \alpha^*)^{1/2}}]^{1/2} .$$

This involves no error for either $\alpha^* = 0$ or $\rho_0 \rightarrow \infty$. In general, the error is negligible when α^* is small, except for ρ_0 about one, that is, τ_0 very small; then our analysis fails anyway because Eq. (5) is invalid. Moreover for any value of ρ_0 the error vanishes at some ϕ_0 between 0 and π , and has opposite signs for larger and smaller angles; this results in a partial error balance in the expression for $\hat{\alpha}^*$. Writing then

$$\rho_0 [1 - (2\alpha^*/\pi)^{1/2} \cos \phi] (1 - \alpha^*)^{-1/2} = \sigma$$

we get

$$\rho_0^2 = \frac{\sigma^2(1 - \alpha^*)}{[1 - (2\alpha^*/\pi)^{1/2} \cos \phi]^2}, \quad (59)$$

$$\sigma \operatorname{Erf}(\ln \sigma)^{1/2} = \rho_m \tau (\pi + \sqrt{\pi} g_2 \tau^2)^{-1/2}. \quad (60)$$

We have dropped the subscripts in both ϕ and τ ; at a given time τ , $\rho_0(\phi, \tau)$ represents now the maximum possible ρ_0 "collected" for each value of ϕ . Equation (59) exhibits an explicit, remarkably simple dependence on α^* and ϕ (the two variables over which further integrations must be carried out). We note that if $\alpha^* = 0$, ρ equals ρ_m at $\tau = 2.80$ for all ϕ ; if $\alpha^* = 0.1$, ρ equals ρ_m at $\tau = 2.80$ for $\phi = 76.4^\circ$ and at smaller (larger) τ for smaller (larger) ϕ . Since the analysis is valid up to distances slightly larger than ρ_m , we may roughly state that our theory begins to break down at $\tau = \tau_m \approx 2.80$.

(c) Nondimensional area $\hat{a}^*(\tau)$

To obtain $\hat{a}^*(\tau)$ we integrate over ρ up to $\rho_0(\phi, \tau)$ or $\rho_A(\phi)$ whichever the lowest, and then integrate over ϕ . In Fig. 4 we again drew schematically the boundaries of region A^* ; we also included several curves of the $\rho = \rho_0(\phi, \sigma)$ family. Notice that there is a value σ_L such that if $\sigma < \sigma_L$, $\rho = \rho_0(\phi, \sigma)$ and $\rho = \rho_A(\phi)$ do not intersect. For $\sigma > \sigma_L$ there are two intersecting points, at angles $\phi_1(\sigma)$ and $\phi_2(\sigma)$; $\phi_1 = \phi_2$ at $\sigma = \sigma_L$. From Eqs. (41) and (59) we get (within an error of less than 1%)

$$\cos \phi_{1,2} = \left(\frac{2}{\pi}\right)^{1/2} \frac{\ln \sigma^2 - 1}{\sigma^2} \frac{1 + \alpha^*}{(\alpha^*)^{1/2}} \pm \left(\frac{1 - \ln \sigma^2 / \sigma^2}{\alpha^*}\right)^{1/2} (\alpha^* - h)^{1/2}, \quad (61)$$

$$h = \left[\frac{\ln \sigma^2}{\sigma^2} - \frac{2}{\pi} \left(\frac{\ln \sigma^2 - 1}{\sigma^2}\right)^{1/2} - \frac{\bar{v}}{\sigma_m^2}\right] \left(1 - \frac{\ln \sigma^2}{\sigma^2}\right)^{-1}; \quad (62)$$

obviously σ_L is given by $\alpha^* = h(\sigma_L)$.

Let us denote $\tau_L = \tau(\sigma_L)$. Then for $\tau < \tau_L$ we get

$$\hat{a}^* = \hat{a}_{(1)}^* - \hat{a}_{(2)}^* \quad (63)$$

$$\hat{a}_{(1)}^* = 2 \int_0^\pi d\phi \int_0^{\rho_0(\phi, \sigma)} \rho d\rho = \int_0^\pi \rho_0^2(\phi, \sigma) d\phi,$$

$$\hat{a}_{(2)}^* = \pi/2 + \rho_p |\cos \phi_p| + \int_{\phi_p}^\pi d\phi \rho_0^2(\phi, \sigma).$$

The area $\hat{a}_{(2)}^*$ corresponds to the probe cross section and to that part of the shaded region lying inside the curve $\rho = \rho_0(\phi, \sigma)$; both were included in $\hat{a}_{(1)}^*$ for simplicity. The point ρ_p, ϕ_p is the intersection of the line $\rho \sin \phi = 1$ ($\pi \geq \phi \geq \pi/2$) and (59). We find

$$\hat{a}_{(1)}^* = \int_0^\pi \frac{\sigma^2 (1 - \alpha^*) d\phi}{[1 - (2\alpha^*/\pi)^{1/2} \cos \phi]^2} = \frac{\pi \sigma^2 (1 - \alpha^*)}{(1 - 2\alpha^*/\pi)^{3/2}}; \quad (64)$$

the dependence of Eq. (64) on α^* amounts to less than 1% and will be neglected,

$$\hat{a}_{(1)}^* = \pi \sigma^2 \quad . \quad (65)$$

Within the same error we find, after a long transformation,

$$\hat{a}_{(2)}^* = \frac{\pi}{2} + \sigma^2 \sin^{-1} \frac{1}{\sigma} + (\sigma^2 - 1)^{1/2} - 2\left(\frac{2\alpha^*}{\pi}\right)^{1/2} \sigma \quad . \quad (66)$$

For $\tau > \tau_L$, $\hat{a}_{(2)}^*$ is unchanged but $\hat{a}_{(1)}^*$ becomes

$$\hat{a}_{(1)}^* = \int_0^{\phi_1(\sigma)} \rho_0^2(\phi, \sigma) d\phi + \int_{\phi_1(\sigma)}^{\phi_2(\sigma)} \rho_A^2(\phi) d\phi + \int_{\phi_2(\sigma)}^{\pi} \rho_0^2(\phi, \sigma) d\phi \quad . \quad (67)$$

The first and third integrals give

$$\begin{aligned} \int_0^{\phi_1} \rho_0^2 d\phi + \int_{\phi_2}^{\pi} \rho_0^2 d\phi &= \frac{(2\alpha^*/\pi)^{1/2} (1 - \alpha^*) \sigma^2 \sin \phi}{(1 - 2\alpha^*/\pi) [1 - (2\alpha^*/\pi)^{1/2} \cos \phi]} \bigg|_{\phi_2}^{\phi_1} \\ &+ \frac{2\sigma^2 (1 - \alpha^*)}{(1 - 2\alpha^*/\pi)^{3/2}} \left(\frac{\pi}{2} + \tan^{-1} \left[\frac{1 + (2\alpha^*/\pi)^{1/2}}{1 - (2\alpha^*/\pi)^{1/2}} \right]^{1/2} \tan \frac{\phi}{2} \right) \bigg|_{\phi_2}^{\phi_1} \end{aligned}$$

and after substitution of $\phi_1(\sigma)$ and $\phi_2(\sigma)$ and some tedious transformations we obtain

$$\int_0^{\phi_1} \rho_0^2 d\phi + \int_{\phi_2}^{\pi} \rho_0^2 d\phi = \sigma^2 [2 \sin^{-1} \left(\frac{h}{\alpha^*} \right)^{1/2} + (\alpha^* - h)^{1/2} h^{1/2} \\ \cdot \left\{ 1 - \frac{4}{\pi} + \frac{12}{\pi \ln \sigma^2} - \frac{8 \bar{v} \sigma^2}{\pi \rho_m^2 \ln \sigma^2} - \frac{2}{\pi (\ln \sigma^2)^2} \right\}] \quad (68)$$

correct up to 1%. When $h \rightarrow \alpha^*$, Eq. (68) approaches $\pi \sigma^2$ and $\phi_1 \rightarrow \phi_2$; thus we recover Eq. (65).

To calculate $\int_{\phi_1}^{\phi_2} \bar{\rho}_A^2 d\phi$ we first obtain an explicit, approximate expression for $\bar{\rho}_A^2(\phi)$ from Eq. (41)

$$\bar{\rho}_A^2 = (\alpha^* \sin^2 \phi + \bar{v}/\rho_m^2)^{-1} \ln [(\alpha^* + \bar{v}/\rho_m^2) \\ \cdot (\alpha^* \sin^2 \phi + \bar{v}/\rho_m^2)^{-1} 1.32 \rho_q^2 \exp \alpha^*] \quad (69)$$

where $\rho_q \equiv \bar{\rho}_A(\pi/2)$ is given by

$$(\alpha^* + \bar{v}/\rho_m^2) \rho_q^2 = \ln \rho_q^2 + \alpha^* + \bar{v}/\rho_m^2$$

The maximum error in Eq. (69) is found to occur at the largest α^* , and amounts to 7.7%. The following points should be noted however:

(1) That error occurs at a certain value of ϕ so that

$\int_{\phi_1}^{\phi_2} \bar{\rho}_A^2 d\phi$ will obviously be more accurate [the more so since the error

in Eq. (69) can have both signs]. (2) After the integration is performed we reverse the change from Eq. (41) to Eq. (69), substantially reducing the error. (3) The error is maximum when the contribution

$\int_{\phi_1}^{\phi_2} \rho_A^{-2} d\phi$ to the final expression for the current is minimum. Carrying out the integration, and substituting ϕ_1 and ϕ_2 from Eq. (61) we finally obtain

$$\begin{aligned} \int_{\phi_1}^{\phi_2} \rho_A^{-2} d\phi = & \frac{2}{\alpha^*} [\pi - 2\sin^{-1} (\frac{h}{\alpha^*})^{1/2}] [(1 + \frac{\bar{v}}{\alpha^* \rho_m^2})^{-1} + \frac{\bar{v}}{3\alpha^* \rho_m^2}] \\ & - \frac{2}{\alpha^*} [h(\alpha^* - h)]^{1/2} \{1 - \frac{2}{\pi} (\frac{\ln \sigma^2 - 1}{\ln \sigma^2})^2\} \\ & + 2(\frac{\sigma^2}{\ln \sigma^2} - 1)^{1/2} (\alpha^* - h)^{1/2} [\ln \sigma^2 (\frac{2}{\pi} - \frac{1}{2}) + 1 \\ & - \frac{2}{\pi} + \frac{4}{\pi \ln \sigma^2} + \frac{C}{\alpha^*} - \frac{2}{3} \frac{\bar{v}}{\rho_m^2} \frac{\ln \sigma^2 - 2/3}{(\alpha^*)^2}] , \end{aligned} \quad (70)$$

$$C = (\ln \sigma^2 - 2) [1 + \frac{2h}{\pi} \{1 - \frac{1}{\ln \sigma^2} + \frac{2}{(\ln \sigma^2)^2}\}] + \frac{\bar{v}}{\rho_m^2} \frac{\sigma^2}{\ln \sigma^2} \frac{\ln \sigma^2 - 2/3}{6} \quad (71)$$

(d) Nondimensional time-averaged current \bar{j}/j_∞

From Eqs. (8), (10) and (11), it follows that

$$\frac{\bar{j}}{j_\infty} = \frac{1}{\tau} \int_0^\tau \frac{j(\tau')}{j_\infty} d\tau' = \frac{\varepsilon}{(8\psi_p)^{1/2} \tau} \int_0^\infty \frac{d\alpha^*}{\alpha} \exp(-\frac{\alpha^*}{\alpha}) \hat{a}^* ; \quad (72)$$

(notice that $m_1(v_1^*)^2/2\kappa T_1 = \beta^*/\beta = \alpha^*/\alpha$). The integral in Eq. (72) may be rewritten as

$$\int_0^\infty d\alpha^* \exp(-\alpha^*/\alpha) \hat{a}^* = \int_0^{h(\sigma)} d\alpha^* \exp(-\alpha^*/\alpha) \hat{a}_{(1)}^* + \int_{h(\sigma)}^\infty d\alpha^* \exp(-\alpha^*/\alpha) \hat{a}_{(1)}^* - \int_0^\infty d\alpha^* \exp(-\alpha^*/\alpha) \hat{a}_{(2)}^*; \quad (73)$$

Eqs. (65) and (67) must be used for the first and second integral respectively. Notice that values of α^* much larger than 0.1 are considered in Eq. (72); if $\alpha \leq 0.1$, however, our estimate of the errors produced by those simplifications based on the assumption $\alpha^* \leq 0.1$ still holds, because as α^* increases beyond 0.1, the increase in those errors is much slower than the exponential decrease of the $\exp(-\alpha^*/\alpha)$ factor. For the last integral in Eq. (73) we find

$$\int_0^\infty d\alpha^* \exp(-\alpha^*/\alpha) \hat{a}_{(2)}^* = \alpha [\pi/2 + \sigma^2 \sin^{-1}(1/\sigma) + (\sigma^2 - 1)^{1/2} - (2\alpha)^{1/2} \sigma]. \quad (74)$$

For the first and second integrals we find (after considerable simplification)

$$\begin{aligned} \int_0^\infty d\alpha^* \exp(-\alpha^*/\alpha) \hat{a}_{(1)}^* &= \alpha \pi \sigma^2 [(1 + 2\eta^{-1}) \operatorname{Erf} \eta^{-1/2} - 2\eta^{-1} + 2(\pi\eta)^{-1/2} \\ &\quad \cdot \exp(-\eta^{-1}) - (\ln \sigma^2)^{-1} \{4(\pi\eta)^{-1/2} \exp(-\eta^{-1}) \operatorname{Erfc} \eta^{-1/2} \\ &\quad - 2\eta^{-1} E_1(2\eta^{-1})\} - \frac{2}{3} \frac{\bar{v}}{c_m} \frac{\sigma^2}{\ln \sigma^2} (\pi\eta)^{-1/2} \exp(-\eta^{-1}) + \frac{1 + 4/\pi}{\pi^{1/2}} \frac{\eta^{1/2}}{\sigma^2} \exp(-\eta^{-1})] \end{aligned} \quad (75)$$

correct up to 1%. We have defined

$$\eta = \frac{\alpha \sigma^2}{\ln \sigma^2} \quad (76)$$

Our results can now be summarized as follows:

$$\bar{j}_{\omega}(\tau, \epsilon, \beta, \psi_p) = \frac{\pi \epsilon \sigma^2}{(8\psi_p)^{1/2} \tau} X(\sigma, \eta, \rho_m) \quad (77)$$

$$X = X_1 + X_2 + X_3 \quad (78)$$

$$X_1 = (1 + 2\eta^{-1}) \operatorname{Erf} \eta^{-1/2} - 2\eta^{-1} + 2(\pi\eta)^{-1/2} \exp(-\eta^{-1})$$

$$- \frac{1 - \sigma^{-2}}{\ln \sigma^2} \{4(\pi\eta)^{-1/2} \exp(-\eta^{-1}) \operatorname{Erfc} \eta^{-1/2} - 2\eta^{-1} E_1(2\eta^{-1})\}, \quad (79)$$

$$X_2 = \left\{ \frac{1 + 4/\pi}{\pi^{1/2}} \frac{\eta^{1/2}}{\sigma^2} - \frac{2}{3} \frac{\bar{v}}{\rho_m^2} \frac{\sigma^2 - 1}{\ln \sigma^2} (\pi\eta)^{-1/2} \right\} \exp(-\eta^{-1}) \quad (80)$$

$$X_3 = -\frac{1}{2\sigma^2} - \frac{1}{\pi} \sin^{-1} \frac{1}{\sigma} - \frac{(1 - \sigma^{-2})^{-1/2}}{\pi\sigma} \left\{ 1 - \left(\frac{2\eta \ln \sigma^2}{\sigma^2 - 1} \right)^{1/2} \right\} \quad (81)$$

where σ , η , ρ_m are given in terms of τ , ϵ , β and ψ_p by

$$\rho_m = (2\psi_p \bar{\delta})^{1/2} / \epsilon \quad (82)$$

$$\sigma \operatorname{Erf}(\ln \sigma)^{1/2} = \rho_m \tau (\pi + \bar{v} g_2 \tau^2)^{-1/2} \quad (83)$$

$$\eta = 2\beta(\sigma^2 - 1)/\psi_p \bar{\delta} \ln \sigma^2 \quad (84)$$

and

$$\bar{v} = 0.80, \bar{g}_2 = 0.75, \bar{\delta}^{-1} = \ln \epsilon^{-1} + Y(\epsilon, \psi_p) \quad (85)$$

where Y is given by Fig. 3.

The instantaneous nondimensional current j/j_∞ can be obtained from Eqs. (77) - (85) and identity

$$\frac{j}{j_\infty} = \frac{d}{d\tau} (\tau \frac{j}{j_\infty})$$

Our emphasis on \bar{j}/j_∞ is based on (1) the greater simplicity of the analysis of $\hat{a}^*(\tau)$ as compared to that of $d\hat{a}^*/d\tau$, and (2) the possibility of using \bar{j}/j_∞ , but not j/j_∞ , in an alternative method of ion temperature determination (see Sec. VI).

For comparison with the approximate numerical computations of Ref. 1, we give in Fig. 5 \bar{j}/j_∞ versus τ for $\epsilon = 10^{-2}$, $\psi_p = 15$ and several values of β , from both those computations and Eqs. (77) - (85). The agreement is good.

X is graphically given in Fig. 6 for $5 < \sigma < 5.78 \times 10^2$ (this is the largest value of σ in the range $\tau \leq 2.80$, $10^{-1} > \epsilon > 10^{-2}$, $10 < \psi_p < 10^2$). We notice that X approaches $X(\sigma \rightarrow \infty)$ quite rapidly: X_2 is never more than 2% of X , X_3 decays quite rapidly as σ increases, and the last term in Eq. (79) is always a small fraction of X_1 . We also note that Fig. 6 does not depend on ρ_m : we used an appropriate,

intermediate value of c_m for each curve in that figure, and the resulting error for other values of c_m is less than $\pm 1\%$.

Finally we point out that to present our final formulae, Eqs. (77) - (85), we substituted $\ln \sigma^2/(1 - \sigma^{-2})$ for $\ln \sigma^2$ in Eqs. (75) and (76); otherwise \bar{j}/j_∞ would go to infinity as $\sigma \rightarrow 1$ ($\tau \rightarrow 0$). Any function $F(\sigma)$ could be used instead of $(1 - \sigma^{-2})$ as long as $\ln \sigma^2/F(\sigma) = O(1)$ at $\sigma = 1$ [the actual choice $F = (1 - \sigma^{-2})$ originates from an analysis of $\phi_1(\sigma)$ and $\phi_2(\sigma)$ at $\alpha^* = 1$]. This arbitrariness justifies the procedure, and is due to the smallness of the contribution of the $\alpha^*/\alpha \gg 1$ range to the integral in Eq. (72). We also wrote $(\sigma^2 - 1)^{1/2}$ for σ in the last term of Eq. (74); otherwise we again would have $\bar{j}/j_\infty \rightarrow \infty$ as $\tau \rightarrow 0$, for $\alpha \neq 0$. The error is due to the simplification leading from Eq. (58) to (59). We stress that these corrections are only required for $\tau \rightarrow 0$; this regime was explicitly excluded from our analysis because of the failure of Eq. (5). Still note that with those corrections $\bar{j}/j_\infty \rightarrow (8/2\psi_p)^{1/2}$ as $\tau \rightarrow 0$, while the true value is $(\pi/2)^{1/2}$ larger, an error of only 20%.

VI - Conclusions

Analytical formulae have been derived for the transient ion current to a long, cylindrical Langmuir probe in a quiescent, collisionless plasma, after a switch in the probe potential from zero to a large negative value ($-eV_p \gg \kappa T_e, \kappa T_i$); the initial ion

distribution function is assumed to be Maxwellian, and the electron Debye length λ_D to be large compared with the probe radius r_p . The transient current density $j(t)$ exhibits a large overshoot before approaching its new steady-state value, $j_\infty = 2^{1/2} \pi^{-1} e N_0 \cdot (-Z_i e V_p / m_i)^{1/2}$. The overshoot is sensitive to the ion temperature and may be used for diagnostic purposes. Assume $-eV_p / \kappa T_e \equiv \psi_p$ and $r_p / \lambda_D \equiv \epsilon$ to be known (from conventional steady state measurements, say) and let \bar{j}/j_∞ be the experimental, non-dimensional, time-averaged current density determined as a function of $\tau \equiv \omega_p t$, where ω_p is the ion plasma frequency. A single point in the curve \bar{j}/j_∞ versus τ can yield the ion temperature: For any couple of values $(\bar{j}/j_\infty, \tau)$, the quantities σ and X , as defined by

$$\sigma \operatorname{Erf}(\ln \sigma)^{1/2} = (2\psi_p \bar{\delta})^{1/2} \epsilon^{-1} \tau (\pi + \bar{\nu} \bar{g}_2 \tau^2)^{-1/2}, \quad (86)$$

$$X = (\bar{j}/j_\infty) (8\psi_p)^{1/2} (\pi\epsilon)^{-1} (\tau/\sigma^2), \quad (87)$$

may be evaluated; the quantity $\eta(\sigma, X)$ may then be obtained from Fig. 6, and, finally, the temperature ratio from

$$\frac{T_i}{Z_i T_e} = \eta \frac{\psi_p \bar{\delta} \ln \sigma^2}{2(\sigma^2 - 1)}. \quad (88)$$

In these equations

$$\bar{v} = 0.80 \quad , \quad \bar{g}_2 = 0.75 \quad , \quad (89)$$

$$\bar{\delta}^{-1} = \ln \epsilon^{-1} + Y(\epsilon, \psi_p) \quad (90)$$

and Y is given by Fig. 3. It should be noted that T_i can be determined even if T_e is unknown. In effect, we can write Eq. (88) as

$$\kappa T_i = - Z_i e V_p \frac{\bar{\delta}}{\sigma^2} \frac{\ln \sigma^2}{2(1 - \sigma^{-2})} \eta \quad .$$

$\psi_p^{1/2}/\epsilon$, and therefore $\bar{\delta}/\sigma^2$, is not a function of T_e . $\bar{\delta}$ does depend on the electron temperature but very weakly: the logarithmic dependence in $\ln \epsilon^{-1}$ is partially balanced by Y . Thus

$$T_i \sim \frac{\ln \sigma^2}{1 - \sigma^{-2}} \eta[\sigma, X(\sigma)]$$

$$\sigma \sim [\bar{\delta}(T_e)]^{1/2} \quad ;$$

in the range of interest of the theory [large σ , $1 < \eta < 100$], no sensible dependence of T_i on T_e is found.

In the process of deriving formulae (86) - (90), certain functions, δ , $\langle v \rangle$ and g_2 , were found to vary between narrow bounds; they were approximated inside a number of integrals by constants $\bar{\delta}$, \bar{v} and \bar{g}_2 as given by Eqs. (89) and (90). It is now possible to

discuss the errors in T_i that may result from these approximations.

For large σ we can write

$$T_i \sim (\pi + \bar{g}_2 \bar{v} \tau^2) \eta[X(\sigma)]$$

$$\sigma \sim \bar{\delta}^{1/2} (\pi + \bar{g}_2 \bar{v} \tau^2)^{-1/2}$$

and for $2 < \eta < 30$ we roughly have

$$X \approx 1 - 0.20 \ln \eta$$

From this and the bounds found earlier for δ we get an estimate of the maximum possible error due to the indeterminacy in $\bar{\delta}$ of less than 10%; when $\tau = 1.5$ a similar result is found for the product $\bar{g}_2 \bar{v}$. We would like to stress that these are very conservative criterions and that the actual error should not be larger than a few percent. We note, however, that the possible error due to $\bar{g}_2 \bar{v}$ grows with τ and reaches 20% at $\tau = 2.80$. Again this is a very conservative estimate, but clearly indicates that the theory should not be used for τ larger than, say, 3.

There are certain experimental conditions that must be met if this method of measuring ion temperature is to be useful. In terms of the four independent parameters τ , ϵ , $\beta \equiv T_i/Z_i T_e$ and ψ_p governing the function \bar{j}/j_∞ , the validity of the present theory is roughly restricted to the range

$$0.1 < \tau < 3 \quad , \quad 10^{-2} \leq \epsilon \leq 10^{-1}$$

$$\beta \leq \frac{\psi_p \bar{\delta}(\epsilon, \psi_p)}{20} \quad , \quad 10 \leq \psi_p \leq 10^2 \quad .$$

Conditions $\epsilon \geq 10^{-2}$ and $\psi_p < 10^2$ could readily be relaxed in the theory. Conditions $\psi_p > 10$ and $20\beta \leq \psi_p \bar{\delta}$ could be somewhat relaxed, but any substantial relaxation would complicate the analysis extraordinarily; these requirements, moreover, are easily met experimentally.

Condition $\epsilon < 10^{-1}$ or, more generally, that ϵ be small, is a more basic one, since when ϵ increases beyond 10^{-1} , say, the ion current becomes rapidly insensitive to T_i and the method ceases to be useful. More specifically, note that for $\eta < 1$, \bar{j}/j_∞ becomes independent of β (see Fig. 6) so that from

$$\eta \sim \frac{\beta}{\epsilon^2} \frac{4}{\ln \sigma^2} \frac{\tau^2}{\pi + 0.6 \tau^2}$$

it appears that for a given ϵ , there is a value of β below which the method can not measure ion temperature; this value is approximately 2×10^{-4} for $\epsilon = 10^{-2}$ and 10^{-2} for $\epsilon = 10^{-1}$. Since a probe of radius $r_p = 10^{-3}$ cm can readily be obtained by conventional methods, requirement $\epsilon \leq 10^{-1}$ is met, at least, by plasmas with $\lambda_D \geq 10^{-2}$ cm.⁵ The requirement is more easily satisfied at higher electron temperatures and lower electron densities.

The restricted range $0.1 < \tau < 3$ creates another experimental limitation. To resolve time on the ion plasma frequency scale it is necessary that the time constant RC of the measuring device be very small compared with the ion plasma period, $2\pi/\omega_p$. Assuming that the resistance R is governed by the requirement that the probe current give rise to a potential $V = y \cdot 10^{-3}$ volts across it, and that the capacitance of the circuit is $C = x \cdot 10^{-12}$ farads, we obtain the condition

$$xy < 3 \cdot 10^{-3} \psi_p^{1/2} \epsilon(\ell/\lambda_D) \quad (\text{No } \lambda_D^3) \quad (91)$$

where ℓ is the length of the probe. For typical instrumentation, y should not be less than about unity, and x would be larger than unity, but of that order. Note that in the above estimate we used j_∞ in computing $R = V/2\pi r_p \ell j$; since $j \gg j_\infty$, Eq. (91) is conservative.

A final consideration to be made is whether the displacement current due to charge accumulation on the probe will mask the true current (particle flux) which is to be measured. Consider the charge balance for the probe. If q_s is the surface charge density, we have

$$j_D \equiv \partial q_s / \partial t = j - j_e - j_T$$

where j and j_e are the ion and electron particle current densities and $2\pi r_p \ell j_T$ is the current measured in the circuit outside the plasma.

j_e may be neglected because it is exponentially small in ψ_p so that our method will be useful as long as j_D/j is small. The displacement current j_D can be rewritten as

$$j_D = \frac{1}{4\pi} \frac{\partial}{\partial t} \left(-\frac{\partial V}{\partial r} \right)_{r=r_p} = \frac{V_p}{4\pi r_p} \frac{\partial}{\partial t} \delta(t)$$

The function $\delta(t)$ was studied in Sec. III; its variation, and therefore j_D , is due to the rearrangement in space charge density following the application of the potential V_p to the probe. Time averaging we find

$$\frac{\bar{j}_D}{j_\infty} = \frac{\pi}{2^{1/2}} \frac{\psi_p^{1/2}}{\epsilon} \frac{\delta(\tau) - \delta(\tau \approx 0)}{\tau};$$

By $\tau \approx 0$ we mean a short time, $\tau \ll 1$, after which the electrons are in equilibrium with the field and $\delta(\tau)$ changes very little; excluding this time (of the order of a few electron plasma periods) in the averaging reduces considerably the value of \bar{j}_D/j_∞ .

A conservative estimate of $[\delta(\tau) - \delta(\tau \approx 0)] \tau^{-1}$ would appear to be $[\delta(\infty) - \delta(\tau \approx 0)] (2\pi)^{-1}$; the calculations of Ref. 1 indicate that the asymptotic approach to the final steady-state value requires several ion plasma periods. With the above estimate we find that \bar{j}_D/j_∞ is just less than one for $\psi_p = 10^2$ and $\epsilon = 10^{-2}$, and goes down with both ψ_p and ϵ^{-1} . Since \bar{j}/j_∞ is typically of order of 10^2 for $\psi_p = 10^2$ and $\epsilon = 10^{-2}$, and varies with ψ_p and ϵ in a way similar to

\bar{j}_D/j_∞ , it appears that \bar{j}_D/\bar{j} is here of order of a few percent. This validates the method.

Even if \bar{j}_D/\bar{j} were not small, the method could be essentially salvaged. We propose the application to the probe of a potential pulse of width $T \approx 3/\omega_p$, instead of the step-wise potential discussed until now. If the time average, now, extends from a few electron plasma periods before the beginning of the pulse up to a few electron plasma periods after the pulse is ended, it is easy to verify that the displacement current cancels out.

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Appendix

To carry out the integration in (49) we rewrote this equation as

$$\tau_0 = 2^{-1/2} \frac{\rho_0}{\rho_m} \int_{\rho_0^{-2}}^1 \frac{ds(1-p)^{-1/2}}{s^{1/2}(-\ln s + \alpha_c^*)^{1/2}} \quad (A1)$$

where $p = p_1 + p_2$ and

$$p_1 = \frac{\alpha^*(1-s) \sin^2 \phi_0}{s(-\ln s + \alpha_c^*)},$$

$$p_2 = \frac{\bar{v}(1-s) \rho_0^2}{\rho_m^2(-\ln s + \alpha_c^*)}.$$

From Eq. (A1) we immediately get

$$\tau_0 = \pi^{1/2} \frac{\rho_0}{\rho_m} \exp\left(\frac{\alpha_c^*}{2}\right) \left[\text{Erf}\left(\ln \rho_0 + \frac{\alpha_c^*}{2}\right)^{1/2} - \text{Erf}\left(\frac{\alpha_c^*}{2}\right)^{1/2} \right] [1 - \bar{p}]^{-1/2} \quad (A2)$$

where \bar{p} is the value of p at some appropriate s between ρ_0^{-2} and 1.

To obtain \bar{p} we expanded the bracket in (A1). In the range of variables considered we found that the p_1 , p_1^2 , p_2 , p_2^2 and p_2^3 terms had to be retained to achieve an error of about 1%. Within an error of (at most) a few percent, all the resulting terms in the expansion of τ_0 could be summed to obtain an expression like (A2) where

$$\bar{p} = \alpha^* \sin^2 \phi_0 + \bar{v}(\rho_0^2/\rho_m^2) g_1(\alpha_c^*),$$

g_1 being given by Eq. (52). The result was Eq. (51).

For the second integral in Eq. (50), the expression inside the square root could be approximated by a polynomial of second degree. The result was Eq. (53).

References

1. S. D. Hester and A. A. Sonin, *Phys. Fluids* 13, 1265 (1970).
2. J. G. Laframboise, University of Toronto Institute for Aerospace Studies Report No. 100 (1966). For additional references see F. F. Chen, in Plasma Diagnostic Techniques, edited by R. H. Huddlestone and S. L. Leonard, (Academic, New York, 1965), Chap. 4.
3. By $\tau \approx 0$ we mean a time $\tau \ll 1$ such that Eq. (5) is already satisfied, while the ions have had no time to respond to the field.
4. It will suffice to notice that $\psi(\tau = \infty, \rho > \rho_{(3)}) > \rho \rho_{(3)}^2 / \rho^2$, as follows from the condition for orbital motion limited current collection.
5. If $\beta \geq 0.1$, and a probe with $r_p = 10^{1/2} \times 10^{-4}$ cm can be obtained, one can move this condition down to $\lambda_D \geq 10^{-3}$ cm.

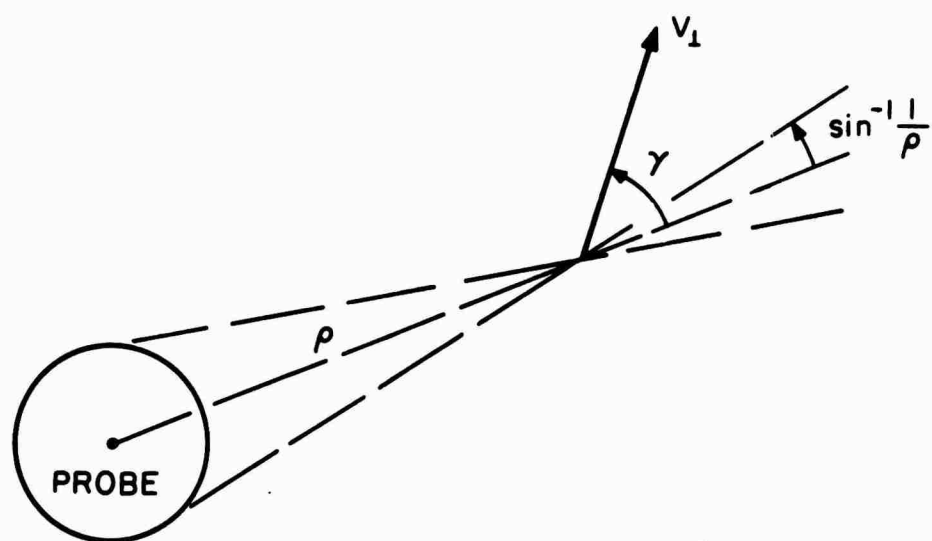


Fig. 1 Diagram for initial ion distribution function.

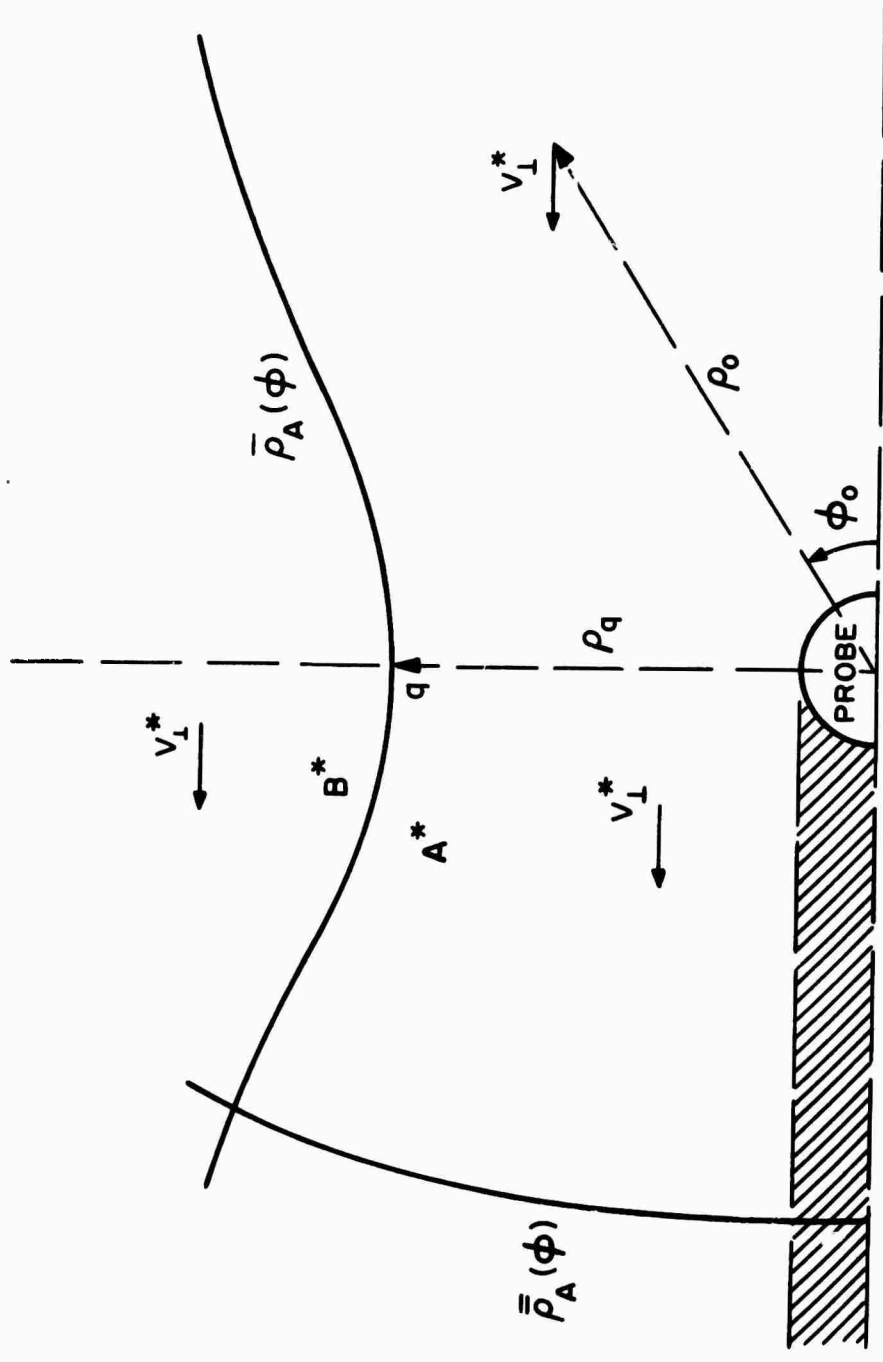


Fig. 2 Diagram showing regions A^* and B^* for simplest initial ion distribution function; density is zero in shaded region, No/Z_i outside it. A^* is the set of initial positions of ions that will be collected at positive times.

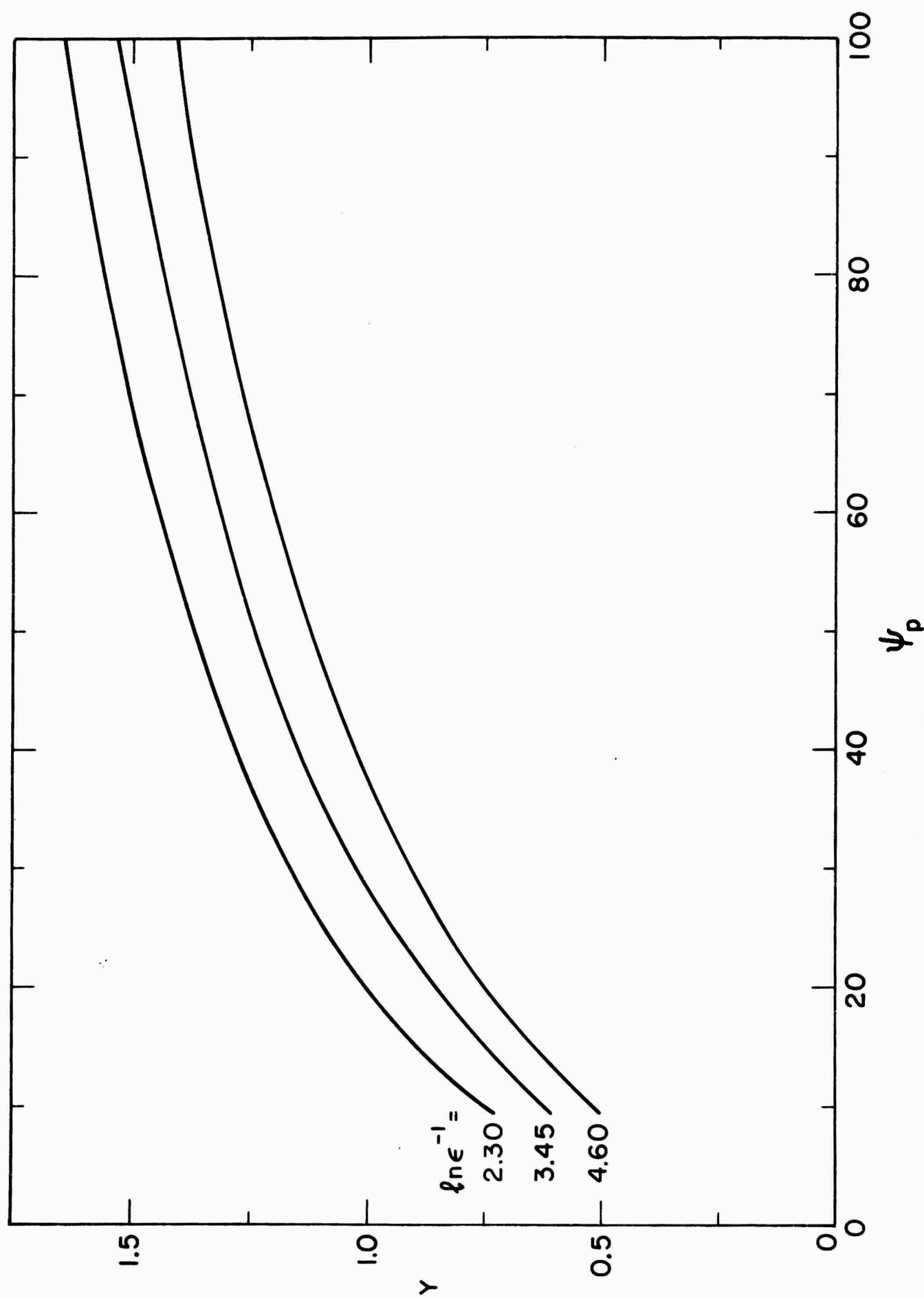


Fig. 3 γ vs. ψ_p and ϵ .

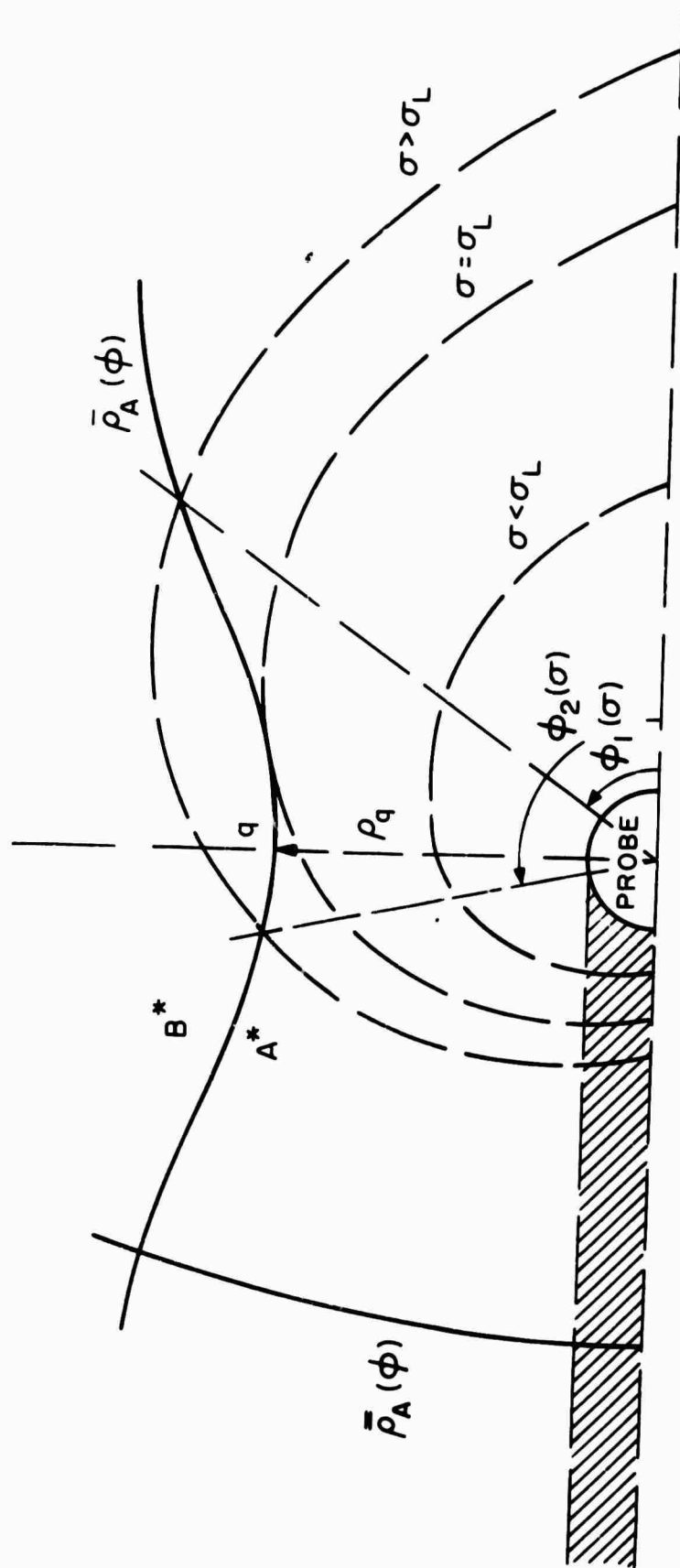


Fig. 4 Diagram for determination of $\hat{a}^*(\tau)$.

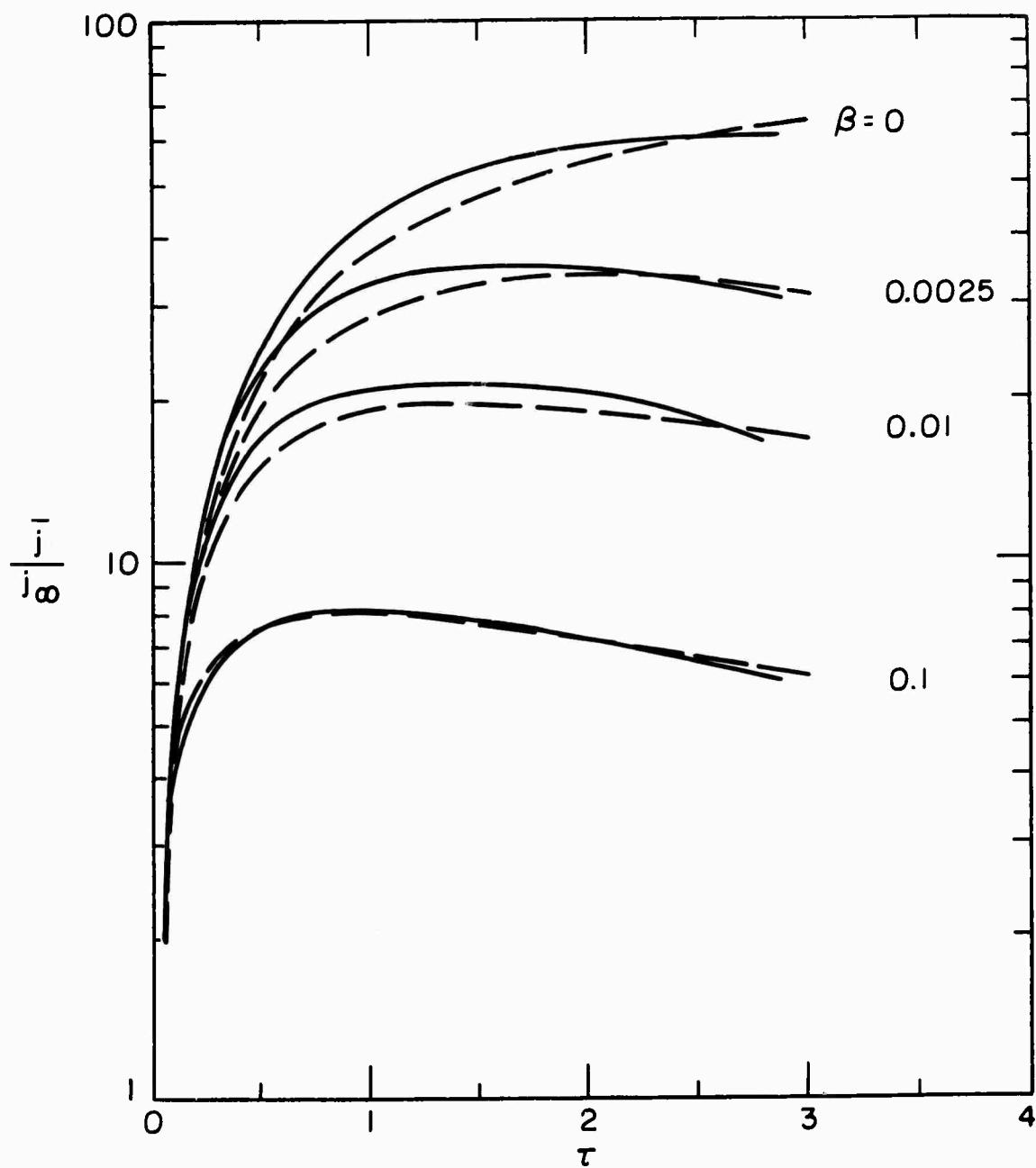


Fig. 5 \bar{j}/j_∞ vs. τ ; $\epsilon = 10^{-2}$, $\psi_p = 15$. Dashed lines, computations of Ref. 1.

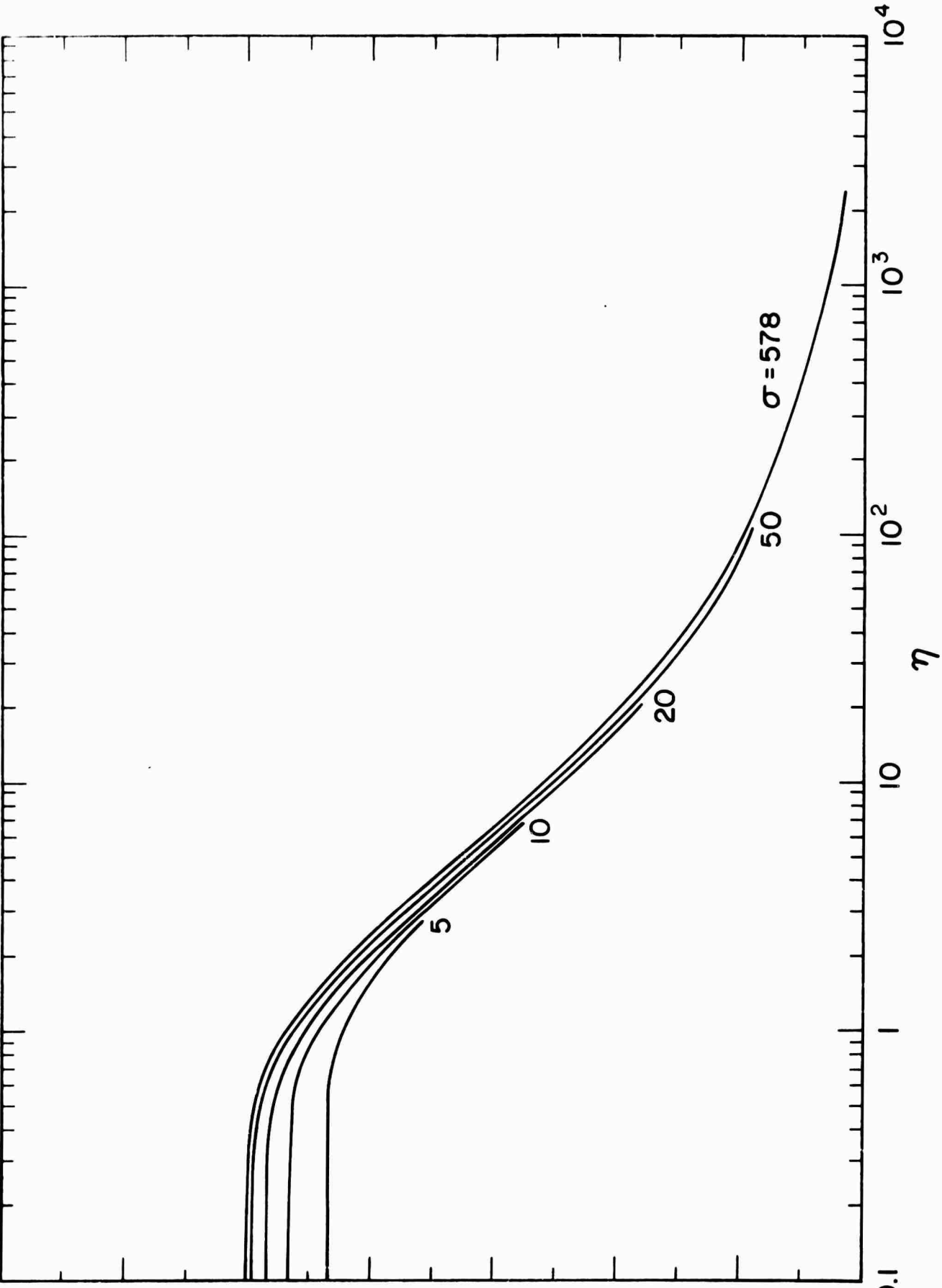


Fig. 6 X vs. η and σ . Curves end at $\alpha \approx 0.3$.